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First degree cohomology of Specht modules over fields of odd prime characteristic

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ABSTRACT

For a field k of odd prime characteristic, we give an easy, but far-reaching sufficient combinatorial condition for a partition λ indicating that the first cohomology $H^1(S_n, S^\lambda)$ of the corresponding Specht module is trivial. Based on this condition, an infinite set of partitions λ is determined, for which the first cohomology groups $H^1(S_n, S^\lambda)$ are one-dimensional.

1. Introduction

In general it is a non-trivial problem to determine the cohomology groups $H^i(G, M) = \mathrm{Ext}^i_{kG}(k, M)$ for a finite group G and a kG-module M, where k is a field of prime characteristic. This is difficult even if G is a largely well understood group like a symmetric group S_n, if M is a prominent kS_n-module like a Specht module, and if $i = 1$. The cohomology groups $H^0(S_n, S^\lambda)$ are known by a result of Gordon James, and it is known that non-trivial 0-cohomology implies non-trivial first cohomology except for $\lambda = (n)$. Further available results about first cohomology concern mostly Specht modules corresponding to hook partitions or two-part partitions. For odd characteristic, David Hemmer described a method that allows (in principle, although it is difficult in practice) to check whether $H^1(S_n, S^\lambda)$ is trivial or not, only by means of combinatorics. The idea is the following:

Let k be a field of odd prime characteristic p. The group $H^1(S_n, S^\lambda) = \mathrm{Ext}^1_{kS_n}(k, S^\lambda)$ parametrizes the extensions of the trivial kS_n-module k with S^λ, where the Ext-group's zero element represents

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the split extension $S^\lambda \oplus k$. Hemmer shows in [3], Theorem 3.2, that every non-split extension can be embedded into the permutation module M^λ. (In characteristic 2 this is different.) That means, $H^1(S_n, S^\lambda) \neq 0$ if and only if there exists a $u \in M^\lambda$ such that the k-subspace $U := \langle S^\lambda, u\rangle_k$ of M^λ is S_n-invariant and indecomposable as kS_n-module with $U/S^\lambda \cong k$ as kS_n-module. Hemmer shows further that such an element u – if it exists – can be constructed as a linear combination of λ-tabloids, whose coefficients satisfy certain relations and do not satisfy others. In this way combinatorics comes into play.

Hemmer formulates several theoretical problems that may be attacked by his method. But if one tries to apply it directly to given partitions, one will mostly end up in calculations that are too involved to carry through. For small n or certain types of two-part partitions (cf. Sections 4 and 5 of [3]) it is a little laborious but possible to prove non-trivial cohomology by this method. But beyond that, it does not seem to provide concrete results at first sight. But appearances are deceiving.

Hemmer points out that his method is closely related to Gordon James' description of $H^0(S_n, S^\lambda)$. Although he explains this relation, it is not obviously reflected in the formulation of his method. Here we will make the connections more evident.

Definition 1.1. For a positive integer m and a non-negative integer a with $p^a \mid m$ and $p^{a+1} \nmid m$ we call $[m]_p := p^a$ the *p-part* of m. We say, the i-th part of λ fulfills the *James-condition* if $\lambda_{i+1} < [\lambda_i + 1]_p$.

(Please note that the original formulation of the James-condition is a little bit different, cf. Theorem 3.8 and Lemma 3.9 (b).) We have $H^0(S_n, S^\lambda) = 0$ if and only if at least one part of λ breaks the James-condition. Now we obtain a similar (but only sufficient) condition for trivial first cohomology:

Theorem 1.2. *Let λ have at least five parts, and let λ_i and λ_j be two parts of λ, except the last part, with $i + 3 \leqslant j$. If both λ_i and λ_j break the James-condition, then $H^1(S_n, S^\lambda) = 0$. (In particular, we have $H^1(S_n, S^\lambda) = 0$ if $p \nmid (\lambda_i + 1)(\lambda_j + 1)$.)*

This will be proved in Remark 3.12, based on Hemmer's method. Theorem 1.2 shows that the strength of Hemmer's method does not lie in proving non-trivial but trivial first cohomology. Theorem 1.2 excludes most partitions from non-trivial first cohomology. The remaining possible candidates are, roughly speaking, partitions with at most four parts or partitions where "almost all" parts are congruent $-1 \pmod p$. A comparatively large example for partitions with non-trivial first cohomology in odd characteristic is given in the following.

Definition 1.3. Let p be an odd prime and $n > 0$ be an integer with $p \mid n$. With $\mathcal{P}_p(n)$ we denote the set of partitions obtained by the following recursion: Start with the Young diagram corresponding to $(n-2, 1^2)$ and add the lowest node with the property that the resulting partition lies in the principal block; repeat this step for each new partition.

Remark 1.4.

(a) Theorem 2.7 will show that the recursion is well defined, that is, there exists always an appropriate addable node.
(b) Besides the recursive description of $\mathcal{P}_p(n)$ it is necessary to know the direct description. The reader is asked to verify that $\mathcal{P}_p(n)$ is given by the following partitions:
  - $(n-2, j, 1)$ for $1 \leqslant j \leqslant p-2$,
  - $(n-2, p-2, 1^2)$,
  - $(n-1, p-2, j, 1)$ for $1 \leqslant j \leqslant p-2$,
  - $(n-1, (p-2)^2, 1^2)$,
  - $(n-1, (p-1)^x, p-2, j, 1)$ for $x \in \mathbb{Z}_{>0}$ and $1 \leqslant j \leqslant p-2$,
  - $(n-1, (p-1)^x, (p-2)^2, 1^2)$ for $x \in \mathbb{Z}_{>0}$.

  (It is helpful to have a graphic image of $\mathcal{P}_p(n)$. For an example with $p = 5$ and $n = 10$, see Fig. 1.)