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On the unit conjecture for supersoluble group algebras

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ABSTRACT

We introduce structure theorems for the study of the unit conjecture for group algebras of torsion-free supersoluble groups. Motivated by work of P.M. Cohn we introduce the class of (X, Y, N)-group algebras KG, and following D.S. Passman we define an induced length function $L : KG \to \mathbb{N} \cup \{-\infty\}$ using the fact that G has the infinite dihedral group as a homomorphic image. We develop *splitting theorems* for (X, Y, N)-group algebras, and as an application show that if $\sigma \in KG$ is a unit, then $L(\sigma) = L(\sigma^{-1})$. We extend our analysis of splittings to obtain a canonical *reduced split-form* for all units in (X, Y, N)-group algebras. This leads to the study of group algebras of virtually abelian groups and their representations as subalgebras of suitable matrix rings, where we develop a determinant condition for units in such group algebras. We apply our results to the *fours group*

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle$$

and show that over any field K, the group algebra $K\Gamma$ has no non-trivial unit of small L-length. Using this, and the fact that L is equivariant under all $K\Gamma$-automorphisms obtained K-linearly from Γ-automorphisms, we prove that no subset of the Promislow set $\mathscr{P} \subset \Gamma$ is the support of a non-trivial unit in $K\Gamma$ for any field K. In particular this settles a long-standing question and shows that the Promislow set is itself not the support of a unit in $K\Gamma$. We then give an introduction to the theory of *consistent chains* toward a preliminary analysis of units of higher L-length in $K\Gamma$. We conclude our work showing that units in torsion-free-supersoluble group algebras are bounded, in that the supports of units and their inverses

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are related through a property (U) and the induced length function L.

1. Introduction

The unit conjecture for group algebras states that if K is a field and G is a torsion-free group, then all units[1] of the group algebra KG are *trivial*; that is, all units are of the form λg for some $\lambda \in K \setminus \{0\}$ and $g \in G$ [11,13,18,19]. The best result to date is entirely group-theoretic, concerning group algebras of unique-product groups [13,14,20]. (A group G is a *unique-product group* if, given any two non-empty finite subsets X and Y of G, there exists an element $g \in G$ having a unique representation of the form $g = xy$ with $x \in X$ and $y \in Y$.) Unique-product groups typify ordered, locally indicable, and right-ordered groups, and for some time it remained an open question whether there exist torsion-free groups that are not unique-product groups. Using small cancellation theory, Rips and Segev [17] gave the first example of a torsion-free group that is not a unique-product group.

For the unit conjecture beyond unique-product groups, it is natural to consider finitely generated, torsion-free, virtually abelian groups; that is, groups with a short exact sequence

$$1 \to H \to G \to G/H \to 1$$

with H abelian and G/H finite. If G/H is cyclic then G is right-orderable, and therefore a unique-product group, so nothing new occurs. The simplest example where G/H is finite non-cyclic is

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle,$$

which satisfies the short exact sequence

$$1 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1.$$

Called the 'fours group', Γ was shown by Passman [13, p. 606] to be torsion-free and non-right orderable. Promislow [16], using a random search algorithm, exhibited a 14-element subset $\mathscr{P} \subset \Gamma$ such that $\mathscr{P} \cdot \mathscr{P}$ has no unique product.[2] Since then, very little progress on the unit conjecture has been made, and, in particular, it has been a long-standing question whether the Promislow set $\mathscr{P}$ could be the support of a unit over some field K.

In this paper we show that the answer is 'no', and in fact prove something slightly stronger, namely that no subset of the Promislow set $\mathscr{P}$ is the support of a non-trivial unit in $K\Gamma$ over any field K. It is of interest to note that our techniques are not simply group-theoretic. Motivated by work of Cohn [3] we introduce the class of (X, Y, N)-group algebras KG, and following Passman [13, Theorem 13.3.7] we define an induced length function $L : KG \to \mathbb{N} \cup \{-\infty\}$ using the fact that G has the infinite dihedral group as a homomorphic image. We develop *splitting theorems* for (X, Y, N)-group algebras, and as an application show that if $\sigma \in KG$ is a unit then $L(\sigma) = L(\sigma^{-1})$. We then extend our analysis of splittings to obtain a canonical *reduced split-form* for all units in (X, Y, N)-group algebras. This leads to the study of group algebras of virtually abelian groups and their representations as subalgebras of suitable matrix rings. This viewpoint allows us to develop a determinant condition for units in such group algebras. We apply our results to the *fours group*

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle$$

[1] *Unit* here means two-sided unit. If K is a field of characteristic 0, then Kaplansky's theorem [8], [13, p. 38] states that every unit in KG is two-sided. The general result for group algebras over fields of characteristic p remains open [2,4,5].

[2] It is an open question as to whether every unique-product group is right orderable.