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Line bundles and curves on a del Pezzo order

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ABSTRACT

Orders on surfaces provide a rich source of examples of noncommutative surfaces. In Hoffmann and Stuhler (2005) [10] the authors prove the existence of the analogue of the Picard scheme for orders and in Chan and Kulkarni (2011) [7] the Picard scheme is explicitly computed for an order on $\mathbb{P}^2$ ramified on a smooth quartic. In this paper, we continue this line of work, by studying the Picard and Hilbert schemes for an order on $\mathbb{P}^2$ ramified on a union of two conics. Our main result is that, upon carefully selecting the right Chern classes, the Hilbert scheme is a ruled surface over a genus two curve. Furthermore, this genus two curve is, in itself, the Picard scheme of the order.

Throughout this paper we assume all objects and maps are defined over an algebraically closed field k of characteristic zero. We denote the dimension of any cohomology group over k by the name of the group written with a non-capital letter for e.g. $\operatorname{ext}^i_A(M,N) := \dim_k \operatorname{Ext}^i_A(M,N)$ and similarly for h^i and hom.

1. Introduction

The study of moduli spaces is an integral part of modern algebraic geometry and representation theory. It is thus very natural, if one is studying noncommutative surfaces, to wish to understand the various moduli spaces that can be associate to them. However, even in the commutative case, let alone the noncommutative one, very few examples have been explicitly computed and the aim of this paper is to slightly remedy this situation.

We focus our attention on studying orders on surfaces which provide a rich source of examples of noncommutative surfaces.

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Definition 1.1. Let X be a normal integral surface. An **order** A on X is a coherent torsion free sheaf of $\mathcal{O}_X$-algebras such that $k(A) := A \otimes_X k(X)$ is a central simple $k(X)$-algebra. X is called the **centre** of A.

See [1] and [4] for an introduction to orders on surfaces.

Since orders are finite over their centres they are in some sense only mildly noncommutative and many classical geometric techniques can be used to study them. In this paper we first fix an order A on $\mathbb{P}^2$ ramified on a union of two conics (these orders play a special role in the Minimal Model Program for orders: see [5]), and study two of its moduli spaces:

(i) the moduli space of line bundles on A (see Definition 3.1), with a fixed set of Chern classes, denoted by **Pic** A, and
(ii) the moduli space of left quotients of A, with a fixed set of Chern classes, denoted by **Hilb** A.

The first moduli space should be thought of as the Picard scheme of A, but one should note that since A-line bundles are only one-sided modules, this is not a group scheme. Borrowing terminology from its commutative counterparts, the second moduli space will be referred to as the Hilbert scheme of A and should be thought of as the space parameterising certain noncommutative curves on A. Not surprisingly, these two moduli spaces are intrinsically linked: in fact we will prove that **Hilb** A is a ruled surface over **Pic** A and that **Pic** A is a genus two curve. Furthermore, by analysing the universal family on **Hilb** A we will show that **Hilb** A exhibits a covering of $\mathbb{P}^2$ with branch locus being two conics and their four bitangents.

The inspiration behind this paper comes from [7] where the authors, Chan and Kulkarni, study the moduli space of line bundles on an order ramified on a smooth quartic. The reader is highly encouraged to read that paper in order to better understand our motivation.

We begin by using the noncommutative cyclic covering trick, described in Section 2.1.1, to construct an order A on $\mathbb{P}^2$ with maximal commutative subalgebra $Y := \mathbb{P}^1 \times \mathbb{P}^1$. Then we study A-modules by noting that they are also naturally $\mathcal{O}_Y$-modules. This allows us to talk about the Chern classes and semistability of A-modules when viewed as $\mathcal{O}_Y$-modules. In particular we will be interested in those A-line bundles with "minimal second Chern class".

We will show that it suffices to consider only two possible first Chern classes: $c_1 = \mathcal{O}_Y(-1,-1)$ with corresponding minimal $c_2 = 0$ and $c_1 = \mathcal{O}_Y(-2,-2)$ with corresponding minimal $c_2 = 2$. The former case will be rather simple and we will prove that the moduli space in that case is just one point. The latter case will be far more interesting and will be our prime focus.

Our main result is:

Theorem 1.2. *Let* **Pic** *A be the moduli space of A-line bundles with* $c_1 = \mathcal{O}_Y(-2,-2)$ *and* $c_2 = 2$ *and* **Hilb** *A – the Hilbert scheme of A, parameterising quotients of A with* $c_1 = \mathcal{O}_Y(1,1)$ *and* $c_2 = 2$*. Then* **Pic** *A is a smooth genus two curve and* **Hilb** *A is a smooth ruled surface over* **Pic** *A. Furthermore,* **Hilb** *A exhibits an* $8:1$ *cover of* $\mathbb{P}^2$*, ramified on a union of* 2 *conics and their* 4 *bitangents.*

In their paper, Chan and Kulkarni had a remarkably similar result concerning the moduli of line bundles with minimal c_2. They also reduced the study of their moduli space of line bundles with minimal second Chern class to two possible first Chern classes. In the first case, the moduli space was a point and in the second case, also a genus two curve.

1.1. Outline of the rest of the paper

We begin by briefly reviewing the relevant theory of orders on surfaces. After this, the rest of the paper is primarily devoted to making sense of, and proving Theorem 1.2. In Section 3 we will define and begin studying line bundles with minimal second Chern classes on our order A. Afterwards, we will introduce the Hilbert scheme of A, compute its dimension and prove that it is smooth. It is here that we will also explore the bizarre covering of $\mathbb{P}^2$ that it exhibits and study its ramification. In the