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KW-sections for Vinberg's θ-groups of exceptional type

Paul Levy

Lancaster University, Department of Mathematics and Statistics, Lancaster LA1 4YF, United Kingdom

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ABSTRACT

Let k be an algebraically closed field of characteristic not equal to 2 or 3, let G be a simple algebraic group of type F_4, G_2 or D_4 and let θ be a semisimple automorphism of G of finite order. In this paper we consider the θ-group (in the sense of Vinberg) associated to these choices; we classify the positive rank automorphisms via Kac diagrams and we describe the little Weyl group in each case. As a result we show that all θ-groups in types G_2, F_4 and D_4 have KW-sections, confirming a conjecture of Popov in these cases.

0. Introduction

Let G be a reductive algebraic group over the algebraically closed field k and let $\mathfrak{g} = \mathrm{Lie}(G)$. Let θ be a semisimple automorphism of G of order m, let $d\theta$ be the differential of θ and let ζ be a primitive m-th root of unity in k. (Thus if k is of positive characteristic p then $p \nmid m$.) There is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \cdots \oplus \mathfrak{g}(m-1) \quad \text{where } \mathfrak{g}(i) = \left\{x \in \mathfrak{g} \,\middle|\, d\theta(x) = \zeta^i x\right\}$$

This is a $\mathbb{Z}/m\mathbb{Z}$-grading of $\mathfrak{g}$, that is $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$ $(i, j \in \mathbb{Z}/m\mathbb{Z})$. Let $G(0) = (G^\theta)^\circ$. Then $G(0)$ is reductive, $\mathrm{Lie}(G(0)) = \mathfrak{g}(0)$ and $G(0)$ stabilizes each component $\mathfrak{g}(i)$. In [24], Vinberg studied invariant-theoretic properties of the $G(0)$-representation $\mathfrak{g}(1)$. The central concept in [24] is that of a *Cartan subspace*, which is a subspace of $\mathfrak{g}(1)$ which is maximal subject to being commutative and consisting of semisimple elements. The principal results of [24] (for $k = \mathbb{C}$) are:

- any two Cartan subspaces of $\mathfrak{g}(1)$ are $G(0)$-conjugate and any semisimple element of $\mathfrak{g}(1)$ is contained in a Cartan subspace;

E-mail address: paul.levy@epfl.ch.

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– the $G(0)$-orbit through $x \in \mathfrak{g}(1)$ is closed if and only if x is semisimple, and is unstable (that is, its closure contains 0) if and only if x is nilpotent;
– let $\mathfrak{c}$ be a Cartan subspace of $\mathfrak{g}(1)$ and let $W_\mathfrak{c} = N_{G(0)}(\mathfrak{c})/Z_{G(0)}(\mathfrak{c})$, the *little Weyl group*. Then we have a version of the Chevalley restriction theorem: the embedding $\mathfrak{c} \hookrightarrow \mathfrak{g}(1)$ induces an isomorphism $k[\mathfrak{g}(1)]^{G(0)} \to k[\mathfrak{c}]^{W_\mathfrak{c}}$;
– $W_\mathfrak{c}$ is a finite group generated by complex (often called pseudo-)reflections, hence $k[\mathfrak{c}]^{W_\mathfrak{c}}$ is a polynomial ring.

In the case of an involution, the decomposition $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ is the symmetric space decomposition, much studied since the seminal paper of Kostant and Rallis [11]. (Many of the results of [11] were generalized to good positive characteristic by the author in [12].) While the theory of θ-groups can in some ways be thought of as an extension of the theory of symmetric spaces, there are certain differences of emphasis. Broadly speaking, one can say that the results here on geometry and orbits are weaker than for symmetric spaces, but the connection with groups generated by pseudoreflections is more interesting.

As outlined in [25, §8.8], for a particularly nice action of a reductive algebraic group G on a vector space V there may exist an affine linear subvariety $W \subset V$, called a Weierstrass section, such that restricting to W induces an isomorphism $k[V]^G \to k[W]$, i.e. such that $W \hookrightarrow V$ is a section for the quotient morphism $V \to V/\!/G = \operatorname{Spec}(k[V]^G)$. In [16] the more specific terminology of *Kostant–Weierstrass*, or *KW*-section was introduced for the case of $G(0)$ acting on $\mathfrak{g}(1)$, because of the similarity to Kostant's slice to the regular orbits in the adjoint representation. This has now become standard terminology in the theory of θ-groups. A long-standing conjecture of Popov [17] is the existence of a KW-section for any θ-group. In characteristic zero, this conjecture was proved by Panyushev in the cases when $G(0)$ is semisimple [15] and when $\mathfrak{g}(1)$ contains a regular nilpotent element of $\mathfrak{g}$ (the 'N-regular' case) [16]. Popov's conjecture is trivially true unless the invariants $k[\mathfrak{g}(1)]^{G(0)}$ are non-trivial, which holds if and only if the dimension of a Cartan subspace is positive; such cases are called *positive rank* θ-groups.

In [13], the results of [24] were extended to the case where k has positive characteristic p and G satisfies the *standard hypotheses*: (A) p is good for G, (B) the derived subgroup G' of G is simply-connected, and (C) there exists a non-degenerate G-equivariant symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to k$. (In fact, most of the results mentioned above hold for all $p > 2$; the standard hypotheses were required for the proof that the little Weyl group is generated by pseudo-reflections.) Moreover, an analysis of the little Weyl group and an extension and application of Panyushev's result on N-regular automorphisms revealed that KW-sections exist for all classical graded Lie algebras in zero or odd positive characteristic [13, Thm. 5.5]. Under the assumption of the standard hypotheses, the problem of the existence of a KW-section can be reduced to the case of a simple Lie algebra or $\mathfrak{gl}_n$, as indicated in [13, §3]. Thus [13, Thm. 5.5] reduces the proof of Popov's conjecture to the following cases: (i) G is of exceptional type and p is good; or (ii) G is simply-connected of type D_4, $p > 3$ and θ is an outer automorphism of G such that θ^3 is inner. Following Vinberg, we refer to all such cases as exceptional type θ-groups.

The automorphisms of finite order of a simple complex Lie algebra were classified by Kac [9]. In Section 2 we give a 'constructive' proof that Kac's classification extends to characteristic p, considering only automorphisms of order coprime to p. (The extension of Kac's results to positive characteristic was outlined by Serre [20]. A proof in the language of Tits buildings has recently been given in joint work of the author and Reeder, Yu and Gross [19]. We include another proof here in order to make this work as accessible as possible, and to describe explicit representatives of each conjugacy class of automorphisms.) Subsequently, we determine in Section 4 the positive rank exceptional θ-groups of types F_4, G_2 and D_4 and describe the corresponding little Weyl groups in Section 5. Our method is a continuation of the method used in [13] to determine the Weyl group for the classical graded Lie algebras in positive characteristic. In particular, given an automorphism θ let T be a θ-stable maximal torus such that $\operatorname{Lie}(T)$ contains a Cartan subspace. Then $\theta = \operatorname{Int} n_w$ where $n_w \in N_{\operatorname{Aut} G}(T)$, and $w = n_w T \in N_{\operatorname{Aut} G}(T)/T$ is either of order m, or is trivial (in which case θ is of zero rank). Using Carter's classification of conjugacy classes in the Weyl group, this approach gives us a relatively straightforward means to determine the positive rank automorphisms and their Weyl groups (see