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Bounding and unbounding higher extensions for SL_2

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ABSTRACT

We analyse the recursive formula found for various Ext groups for $SL_2(k)$, k a field of characteristic p, and derive various generating functions for these groups. We use this to show that the growth rate for the cohomology of $SL_2(k)$ is at least exponential. In particular, $\max\{\dim \operatorname{Ext}^i_{SL_2(k)}(k, \Delta(a)) \mid a, i \in \mathbb{N}\}$ has (at least) exponential growth for all p. We also show that $\max\{\dim \operatorname{Ext}^i_{SL_2(k)}(k, \Delta(a)) \mid a \in \mathbb{N}\}$ for a fixed i is bounded.

Introduction

A very general open problem in the characteristic p representation theory of algebraic groups is to determine the higher extension groups $\operatorname{Ext}^q_G(M, N)$ where M, N are Weyl modules, or simple modules (or more generally if possible). In [7] the third author found recursive formula for many different Ext groups for modules in $SL_2(k)$, k an algebraically closed field of characteristic p. But no closed formula were found. More recently there has been interest in finding upper bounds for the dimensions of Ext groups (see for example [8]). Work of [10] applied the results of [7] to show that the growth rate of $\operatorname{Ext}^i_{SL_2(k)}(k, L)$, taken over all L a simple module and $i \in \mathbb{N}$ is at least exponential. Several years ago, just after [7] had been done, we had found some generating functions for the dimensions of Ext groups for Weyl modules, to investigate how large these could be, (mentioned in [2, Section 6]). The recent work of Parshall and Scott in [8] and [9], and of Stewart [10], has encouraged us to polish this work up to explore further questions raised by these authors. In particular, using our generating functions, we have got an analogue of the exponential growth found by Stewart [10]

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for all primes, but using Weyl modules rather than simples. We also show that when we fix i that $\max\{\dim \mathrm{Ext}^i_{\mathrm{SL}_2(k)}(k, \Delta(a)) \mid a \in \mathbb{N}\}$ is bounded (see Section 7).

For prime 2 we have an explicit formula for the dimension of $\mathrm{Ext}^n_{\mathrm{SL}_2(k)}(\Delta(0), \Delta(a))$. When a is odd this is zero by block considerations, so let $a = 2d$. We show that this is equal to the number of partitions $(b_0, b_1, \dots, b_n)$ such that $\sum_i 2^{b_i} = d + 1$ (see Corollary 3.2.2). These are compatible with Stewart's results.

For $p > 2$ the situation is more complicated. Our generating function $G(s)$ can be written as $\sum_{n \geqslant 0} z^n h_n(s)$ with $h_n(s)$ a power series, and the coefficient of s^d in $h_n(s)$ is the dimension of $\mathrm{Ext}^n_{\mathrm{SL}_2(k)}(\Delta(0), \Delta(2d))$. We have a recursion for $sh_n(s)$, given in 4.5 and an algorithm which is described in 4.7. Exponential growth is established for $p = 2$ in Proposition 6.1.1 and for odd primes in Lemma 5.5.2 together with Lemma 5.6.2. That is, for a fixed prime, if we let d vary with n, then the dimensions of $\mathrm{Ext}^n_{\mathrm{SL}_2(k)}(\Delta(0), \Delta(2d))$ grow exponentially.

1. Preliminaries

1.1. Notation

We first briefly review some of the notation and definitions that we will use in this paper. The reader is referred to [4] for further information. We let $G = \mathrm{SL}_2(k)$ where k is an algebraically closed field of characteristic p, and $\mathrm{F} : G \to G$ the corresponding Frobenius morphism. We may "twist" G-modules via this morphism. We let X^+ be the set of dominant weights which may be identified with $\mathbb{N}$, the non-negative integers.

For $\lambda \in X^+$, let k_λ be the one-dimensional module for B a suitable Borel, which has weight λ. We define $\nabla(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$. This module has character given by Weyl's character formula and has simple socle $L(\lambda)$, the irreducible G-module of highest weight λ. In the case of SL_2 all simples are known via Steinberg's tensor product theorem. If we let E be the 2-dimensional natural module for $\mathrm{SL}_2(k)$, then $\nabla(\lambda) = S^\lambda E$, the λ-th symmetric power of E. We will also use Weyl modules $\Delta(\lambda)$ which for our purposes can be either thought as divided powers, so $\Delta(\lambda) = D^\lambda E$ or as duals of induced modules: $\Delta(\lambda) = \nabla(\lambda)^*$, where $*$ is the usual k-linear dual.

The category of rational G-modules has enough injectives and so we may define $\mathrm{Ext}^*(-,-)$ as usual by using injective resolutions.

1.2. Background

Past work of the third author [5,6], was concerned with finding explicit bounds on the global dimension of the Schur algebra associated to polynomial modules for $\mathrm{GL}_n(k)$. Of course it was known that such a bound should exist as the category of G-modules with bounded highest weight is an example of what is now known as a high weight category (or equivalently, is the module category of a quasi-hereditary algebra) and as such has finite global dimension. The work of [6] showed that for any algebraic group that

$$\mathrm{Ext}^m_G\big(L(w\cdot\lambda), L(v\cdot\lambda)\big) = \mathrm{Ext}^m_G\big(\nabla(w\cdot\lambda), \Delta(v\cdot\lambda)\big) = \begin{cases} 0 & \text{if } m > l(w) + l(v), \\ k & \text{if } m = l(w) + l(v) \end{cases}$$

where λ is in the interior of the fundamental alcove, $w, v \in W_p$, the affine Weyl group acting via the dot action on the dominant weights and $l : W_p \to \mathbb{N}$ is the usual length function on (the Coxeter group) W_p. We also have:

$$\mathrm{Ext}^m_G\big(\Delta(w\cdot\lambda), \Delta(v\cdot\lambda)\big) = \begin{cases} 0 & \text{if } m > l(v) - l(w), \\ k & \text{if } m = l(v) - l(w) \end{cases}$$

with the same notation as above.