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Idempotents in ring extensions ☆

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ABSTRACT

The aim of the paper is to study idempotents of ring extensions $R \subseteq S$ where S stands for one of the rings $R[x_1, x_2, \ldots, x_n]$, $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$, $R[[x_1, x_2, \ldots, x_n]]$. We give criteria for an idempotent of S to be conjugate to an idempotent of R. Using our criteria we show, in particular, that idempotents of the power series ring are conjugate to idempotents of the base ring and we apply this to give a new proof of the result of P.M. Cohn (2003) [4, Theorem 7] that the ring of power series over a projective-free ring is also projective-free. We also get a short proof of the more general fact that if the quotient ring R/J of a ring R by its Jacobson radical J is projective-free then so is the ring R.

1. Introduction

By the ring extension S of an associative unital ring R we mean, in this article, one of the following rings: the polynomial ring $R[x_1, \ldots, x_n]$ in finite number of commuting indeterminates $x_1, \ldots, x_n$, the Laurent polynomial ring $R[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}]$ and the power series ring $R[[x_1, \ldots, x_n]]$. The aim of the paper is to study relations between idempotents of R and those of S. One of the motivations of our study is the Quillen and Suslin solution of Serre's problem which says that every finitely generated projective module over a polynomial ring $K[x_1, \ldots, x_n]$, where K is a field, is free (cf. [8] for more details). Since any finitely generated projective module is associated with an idempotent of a matrix ring, the above result can be translated in terms of idempotents as follows: every idempotent $e^2 = e \in M_l(K)[x_1, \ldots, x_n]$ is conjugate to an idempotent in the base ring $M_l(K)$.

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$E(R)$ will denote the set of all idempotents of a ring R. At the beginning, we present necessary and sufficient conditions for the equality $E(S) = E(R)$ (Corollary 6). As a byproduct of our investigations we obtain a short proof of a result of H. Bass on idempotents in commutative group rings.

Two elements a, b in a ring R are said to be *conjugate* if there exists an invertible element $u \in R$ such that $b = uau^{-1}$. We provide, in Corollary 11, a sufficient condition for two idempotents in a ring to be conjugate. With the help of this condition we show, in Theorem 13, that any idempotent of $R[[x_1, \dots, x_n]]$ is conjugate to an idempotent of R. We also study situations when a similar result holds for a polynomial ring.

Let us recall a few definitions: a ring R is 2-*primal* if the set of its nilpotent elements is exactly the prime radical of R. A ring R is *abelian* if all idempotents of R belong to its center. It is well known that reduced rings are abelian (cf. [6]). We show that every idempotent of a polynomial ring $R[x]$ is conjugate to an idempotent of R in the following cases: R is abelian, R is the matrix ring $M_n(A)$, where A is either a division ring or a polynomial ring $K[x_1, \dots, x_m]$ over a field K. It is also shown that any idempotent of degree one in $R[x]$ is conjugate to an idempotent of R. Based on a result of Ojanguren and Sridharan we give an explicit example of a polynomial e of degree two with coefficients in $R = M_2(\mathbb{H}[y])$, which is an idempotent of $R[x]$ not conjugate to any idempotent of R. In fact, there are uncountably many nonconjugate such idempotents.

We also show that, for any ring R, the semicentral idempotents of $R[x]$ are conjugate to idempotents of R (Theorem 18).

A ring R is *projective-free* if every finitely generated left (equivalently right) R-module is free of unique rank. As a consequence of our investigations, we give new short proofs of a series of classical results, namely, a theorem of P.M. Cohn saying that the projective-free property lifts up from R/J to R, where J is the Jacobson radical of R (Theorem 21(a)); a particular case of a result of I. Kaplansky which says that local rings are projective-free (Theorem 21(b)); another theorem of P.M. Cohn stating that if R is projective-free then so is $R[[x]]$ (Theorem 22) and a theorem of G. Song and X. Guo saying that two idempotents in a ring are equivalent if and only if they are conjugate (Corollary 20).

2. Idempotents

We begin with the following elementary result (cf. [1, Proposition 2.5]).

Lemma 1. *Let R be a ring and $e(x) = \sum_{i=0}^{\infty} e_i x^i \in R[[x]]$ be an idempotent. If $e_0 e_i = e_i e_0$, for every $i \geqslant 1$, then $e(x) = e_0$. In particular, if R is abelian, then $E(R[[x]]) = E(R[x]) = E(R)$.*

Proof. It is clear that e_0 is an idempotent of R. Assume that $e(x) \neq e_0$ and let $k > 0$ be the least index such that $e_k \neq 0$. Comparing the degree k coefficients of $e(x)^2$ and $e(x)$ we get $2e_0 e_k = e_k$. Multiplying this equality by e_0 we obtain $e_0 e_k = 0$ and hence also $e_k = 0$. This contradiction yields the result. □

The above observation is also contained in Lemma 8 of [5]. As a first application we prove the following result which can also be obtained by combining Propositions 2.4 and 2.5 in [1] and Lemma 1.7 in [2].

Proposition 2. *Let S_n denote one of the following rings $R[x_1, \dots, x_n]$, $R[[x_1, \dots, x_n]]$ and $R[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}]$. If e is a central idempotent of S_n, then $e \in R$.*

Proof. Lemma 1 gives the result for rings $R[x]$ and $R[[x]]$. Since the element $1 - x$ is invertible in $R[[x]]$, the R-monomorphism $\phi : R[x] \to R[[x]]$ which sends x onto $1 - x$ has a natural extension to an R-monomorphism $\phi : R[x^{\pm 1}] \to R[[x]]$ given by $\phi(x^{-1}) = (1 - x)^{-1}$. Let $e \in R[x, x^{-1}]$ be a central idempotent. Since e commutes with elements of R the same property holds for the idempotent $\phi(e) \in R[[x]]$. This implies that $\phi(e) \in R$ and $e \in R$ follows. Hence the result holds for $R[x^{\pm 1}]$. Now it is standard to complete the proof by induction on n. □

Remark 3. As in the above proposition, let S_1 denote either $R[x]$, $R[x, x^{-1}]$ or $R[x]$. It is easy to check that if δ is a derivation of a ring R then, for any central idempotent $e \in R$, we have $\delta(e) = 0$. Thus,