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Inversion formula for the growth function of a cancellative monoid

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ABSTRACT

Let $(M, \deg)$ be a cancellative monoid M equipped with a degree map $\deg: M \to \mathbb{R}_{\geqslant 0}$, and let $P_{M,\deg}(t) := \sum_{u \in M/\sim_l} t^{\deg(u)}$ be its generating series (here, $u \sim_l v$ for $u, v \in M$ if $u \mid_l v$ and $v \mid_l u$, see Section 2, Definition), called the *growth function* of $(M, \deg)$. In the present paper, we introduce the *skew-growth function* $N_{M,\deg}(t)$ of $(M, \deg)$, and prove the *inversion formula*

$$P_{M,\deg}(t) \cdot N_{M,\deg}(t) = 1$$

where the skew-growth function $N_{M,\deg}(t)$ is a signed generating series

$$N_{M,\deg}(t) := 1 + \sum_{T \in \mathrm{T}(M,I_0)} (-1)^{\#J_1+\cdots+\#J_n-n+1} \sum_{\Delta \in \mathrm{mcm}(T)} t^{\deg(\Delta)}$$

of minimal common multiples $\Delta \in \mathrm{mcm}(T)$ of towers $T \in \mathrm{T}(M, I_0)$ associated with the minimal set I_0 of $M/\sim$.
If $M = \mathbb{Z}_{>0}$ is the multiplicative monoid of positive integers with $\deg = \log$, then $P_{M,\deg}(\exp(-s))$ is Riemann zeta function and the inversion formula turns out to be the Euler product formula. We discuss more examples on the behavior of the analytic function $N_{M,\deg}(\exp(-s))$ on the boundary line $\Re(s) =$ abscissa of absolute convergence of $P_{M,\deg}(\exp(-s))$.

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Contents

1. Introduction

Let M be a monoid, i.e. a semi-group with the unit 1, and let $\deg : M \to \mathbb{R}_{\geqslant 0}$ be a discretely valued degree map on M (see Section 4). Then the (spherical) growth function of M with respect to deg is defined as the generating series:

$$P_{M,\deg}(t) := \sum_{[u]\in M/\sim} t^{\deg([u])},$$

where the equivalence $u \sim v$ for $u, v \in M$ means the mutual left divisibility relations $u \,|_l\, v$ and $v \,|_l\, u$, and $[u]$ denotes the equivalence class of u (see Section 2, Definition). Even though the definition of the growth function is a simple generalization of that for the case of groups, not much work seems to be available except for some studies of special cases or some general works on language [A-N,Bra, Bri-S,Bro,C,C-F,D,G,G-P,I1,Ku,P,S1,S2,S3,S4,S-I], and we know little about the general nature of these functions. The purpose of the present paper is to give a new approach to the subject by giving a presentation of the inversion function $\frac{1}{P_{M,I}(t)}$ by a certain *"skew-growth function"*[1] *of common multiple sets in the monoid M*.

Let us explain the inversion function in the most naive case studied in [A-N] and in [S2,S3]. Let M be a monoid generated by a finite set I satisfying some positive homogeneous relations. The monoid naturally admits an integer-valued degree map by giving weight 1 to each generator in I. Suppose, further, that M is cancellative (see Section 2) and that any subset J of I either admits a unique least right common multiple Δ_J or has no common multiple in M (typically the case for Artin monoids [Bri-S,D]). Then the inversion function $P_{M,I}(t)^{-1}$ is given by

$$N_{M,I}(t) := \sum_{J\subset I} (-1)^{\#J} t^{\deg(\Delta_J)}$$

where the summation index J runs over all subsets of I such that there exists in M the least right common multiple Δ_J of all elements $u \in J$.[2]

That is, *the inversion function of the growth function for this class of monoids is given by the skew-growth function $N_{M,I}(t)$ of the set $\{\Delta_J\}_{J\subset I}$ of all least right common multiples for subsets J of the generator set I.* However, in general, a monoid may not admit a least right common multiple Δ_J for a given set $J \subset M$. More precisely, even if there exist some right common multiples of J, there may not exist

[1] By a *skew-growth function*, we mean a suitably signed generating series.

[2] In this case, $N_{M,I}(t)$ is a polynomial. A proof of the inversion formula is given by the fact that the coefficients of $N_{M,I}(t)$ give the recursion relation on the sequence of the coefficients of $P_{M,\deg}(t)$. This fact will be generalized in Section 5 of the present paper. Zero loci of the polynomial $N_{M,I}(t)$ plays a quite important role in the study of limit functions [S1,S2,S3,S4, K-T-Y], which motivated the author to the present work.