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Commutative group rings with von Neumann regular total rings of quotients

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ABSTRACT

Let R be a commutative ring and let G be an abelian group. We show that if G is either torsion free or R is uniquely divisible by the order of every element of G, then the von Neumann regularity of the total ring of quotients of R ascends to the total ring of quotients of RG. Examples are given to show that the converse does not hold. These results are applied in the group ring setting to explore a number of zero divisor controlling conditions, such as being a PF or a PP ring as well as a number of Prüfer conditions, such as being an arithmetical, a Gaussian, or a Prüfer ring.

1. Introduction

Let R be a commutative ring, let G be an abelian group, and denote by RG the group ring of G over R. In this article we explore a number of zero divisor and Prüfer conditions in the group ring setting.

In 1940, Higman [14] characterized when RG is a domain. A classical result in [16] addresses the case when RG is reduced. Subsequent efforts to control zero divisors of commutative group rings have shifted towards homological conditions that were recently linked to Prüfer conditions. Two such notions are the PP (Principal Projective) and the PF (Principal Flat) properties of rings. Both PP and

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PF rings are locally domains [9,20]. These conditions were introduced in the 1960s by Hattori [13] and Endo [5] primarily to develop a torsion theory for modules. They were extensively studied in the 1970s in conjunction with another zero divisor controlling condition, namely the property of having von Neumann regular total ring of quotients [6,15,20,23,25]. More recently, the PP, PF, and von Neumann regular total ring of quotients conditions have been used to explore the extensions of the notion of a Prüfer domain to rings with zero divisors [2,10–12,19]. These extensions of the Prüfer domain notion include semihereditary rings, rings of weak global dimension one, arithmetical rings, Gaussian rings, locally Prüfer rings, and Prüfer rings. The PP and PF conditions have also been investigated recently in other contexts, e.g. [1,26,17]. In particular [22,26] began to address these conditions in the group ring setting.

In Section 2 we determine conditions under which the von Neumann regularity of the total ring of quotients of R ascends to the total ring of quotients of RG. This happens when G is either torsion free or R is uniquely divisible by the order of every element of G (Theorem 2.3). We also provide examples (Example 2.4) to show that descent of this property from RG to R does not necessarily hold under either condition.

Section 3 explores the PP and PF conditions, while Section 4 explores the six Prüfer conditions mentioned above in the group ring setting. Many of the results and examples in both of these two sections make use of the ascent of von Neumann regularity of the total ring of quotients results obtained in Section 2.

In Section 3 we show that if G is torsion free, then RG is a PF (respectively, PP) ring if and only if R is a PF (respectively, a PP) ring (Theorem 3.3). In general, if RG is a PF (respectively, PP) ring then either G is torsion free or R is uniquely divisible by the order of every element of G (Proposition 3.5). But, in general, neither the PF nor the PP conditions ascend from R to RG (Example 3.6).

In Section 4 we show that if G is a torsion free or a mixed abelian group, then the six Prüfer conditions are equivalent to each other and also equivalent to R being von Neumann regular and $\operatorname{rank} G = 1$ (Theorem 4.3(i)). If G is torsion and R is uniquely divisible by the order of every element of G, the equivalence of the six Prüfer conditions, and analogous statements on R and G, require the additional assumption of von Neumann regularity of the total ring of quotients of R (Theorem 4.3(ii)). An example is provided (Example 4.4) that shows that this assumption is necessary.

Throughout the paper R will always denote a commutative ring with identity, G will denote an abelian group written multiplicatively, and $Q(R)$ will denote the total ring of quotients of R. Also $\mathbb{Q}$ will denote the rational numbers, and $\mathbb{C}$ will denote the complex numbers.

2. Von Neumann regularity of the total ring of quotients

In this section we explore ascent and descent of the von Neumann regularity condition between the total ring of quotients of R, $Q(R)$, and the total ring of quotients of RG, $Q(RG)$. The following condition, which appears often in investigations involving homological properties of group rings, links R and G via a divisibility property.

Definition 2.1. Let R be a commutative ring, and let G be an abelian group. R is said to be uniquely divisible by the order of every element of G if for every g in G of finite order n, n divides every element $r \in R$, and if for $r \in R$, we have $r = ns = nt$ for some $t, s \in R$, then $s = t$.

This condition is equivalent to asking that every prime number p, which is the order of an element g in G, be a unit in R. For $x = \sum x_g g \in RG$, let $\operatorname{supp} x = \{g \in G\colon\ x_g \neq 0\}$.

Lemma 2.2. *Let G be an abelian group. Write $G = \varinjlim G_i$ where G_i are the finitely generated subgroups of G ordered by inclusion. Then $Q(RG) = \varinjlim Q(RG_i)$ and $\text{w.gl.dim}\, Q(RG) \leqslant \sup\{\text{w.gl.dim}\, Q(RG_i)\}$.*

Proof. Consider $f/g \in Q(RG_i)$. $RG_i \subseteq RG_j$, where $i < j$, is a free extension and g is not a zero divisor in RG_i. It follows that g is not a zero divisor in RG_j. Therefore $Q(RG_i) \subseteq Q(RG_j)$ and $\varinjlim Q(RG_i) = \bigcup Q(RG_i)$ exists. Since $RG_i \subset RG$ is a flat extention, g is also not a zero divisor in RG. This