Journal of Algebra 363 (2012) 8–13

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the annihilators of local cohomology modules

Kamal Bahmanpour [a], Jafar A'zami [b,*], Ghader Ghasemi [b]

[a] Department of Mathematics, Ardabil Branch, Islamic Azad University, P.O. Box 5614633167, Ardabil, Iran
[b] Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran

ARTICLE INFO

Article history:
Received 5 March 2011
Available online 4 May 2012
Communicated by Bernd Ulrich

MSC:
13D45
14B15
13E05

Keywords:
Associated primes
Krull dimension
Local cohomology

ABSTRACT

Let $(R,\mathfrak{m})$ be a commutative Noetherian complete local ring, M a non-zero finitely generated R-module of dimension $d \geqslant 1$, and $T_R(M) := \bigcup\{N\colon\ N \leqslant M \text{ and } \dim N < \dim M\}$. In this paper we calculate the annihilator of the top local cohomology module $H^d_{\mathfrak{m}}(M)$. More precisely, we show that $0 :_R H^d_{\mathfrak{m}}(M) = 0 :_R M/T_R(M)$. Moreover, for every positive integer n, we calculate the radical of the annihilator of $H^n_{\mathfrak{m}}(M)$. More precisely, we prove that if $H^n_{\mathfrak{m}}(M)$ is not finitely generated then $\operatorname{Rad}(0 :_R H^n_{\mathfrak{m}}(M)) = \bigcap_{\mathfrak{p}\in S}\mathfrak{p}$, where

$$S = \bigl\{\mathfrak{p} \in \operatorname{Spec} R\colon\ \bigl(H^{n-1}_{\mathfrak{p}}(M)\bigr)_{\mathfrak{p}} \neq 0 \text{ and } \dim R/\mathfrak{p} = 1\bigr\}.$$

1. Introduction

Throughout this paper, let R denote a commutative Noetherian local ring (with identity) and I an ideal of R. For an R-module M, the ith local cohomology module of M with respect to I is defined as

$$H^i_I(M) = \varinjlim_{n\geqslant 1} \operatorname{Ext}^i_R\bigl(R/I^n, M\bigr).$$

For an Artinian R-module A we denote by $\operatorname{Att}_R A$ the set of attached prime ideals of A. For each R-module L, we denote by $\operatorname{Assh}_R L$ (resp. $\operatorname{mAss}_R L$) the set $\{\mathfrak{p} \in \operatorname{Ass}_R L\colon\ \dim R/\mathfrak{p} = \dim L\}$ (resp. the set of minimal primes of $\operatorname{Ass}_R L$). Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R\colon\ \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set

* Corresponding author.
E-mail addresses: bahmanpour.k@gmail.com (K. Bahmanpour), jafar.azami@gmail.com, azami@uma.ac.ir (J. A'zami), ghghasemi@uma.ac.ir (G. Ghasemi).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.03.026

$\{x \in R\colon\ x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. We refer the reader to [4] or [2] for more details about local cohomology. The main results of this paper are the following:

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and M be a non-zero finitely generated R-module of dimension d. Then*

$$0 :_R H^d_{\mathfrak{m}}(M) = 0 :_R M/T_R(M),$$

where, $T_R(M) := \bigcup\{N\colon\ N \leqslant M$ and $\dim N < \dim M\}$.

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a complete Noetherian local ring and $n \geqslant 1$ be an integer. Let M be a non-zero finitely generated R-module such that $\dim M \geqslant 1$. Set $J := \operatorname{Rad}(0 :_R H^n_{\mathfrak{m}}(M))$ and $S = \{\mathfrak{p} \in \operatorname{Spec} R\colon\ (H^{n-1}_{\mathfrak{p}}(M))_{\mathfrak{p}} \neq 0$ and $\dim R/\mathfrak{p} = 1\}$. Then the following statements hold*:

(i) *If $H^n_{\mathfrak{m}}(M)$ is not finitely generated, then $J = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.*
(ii) *If $H^n_{\mathfrak{m}}(M)$ is non-zero and finitely generated, then $J = \mathfrak{m}$.*

Theorem 1.3. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then $0 :_R H^d_{\mathfrak{m}}(R) = T_R(R)$.*

For any unexplained notation and terminology we refer the reader to [2] and [5].

2. The results

To prove the main results of this paper, we need the following proposition.

Proposition 2.1. *Assume that $(R, \mathfrak{m})$ is a Cohen–Macaulay local (Noetherian) ring of dimension d and for every $\mathfrak{p} \in \operatorname{Ass}_R R$ the zero-dimensional local ring $R_{\mathfrak{p}}$ is Gornestein. Let M be a finitely generated R-module such that $\operatorname{Ass}_R M \subseteq \operatorname{Ass}_R R$. Then there is an exact sequence $0 \to M \to \bigoplus_{i=1}^n R$ for some positive integer n.*

Proof. See [3, Theorem 3.5]. □

The following result follows from Proposition 2.1.

Corollary 2.2. *Assume that $(R, \mathfrak{m})$ is a Gorenstein local (Noetherian) ring and I be an ideal of R. Then the following statements are equivalent*:

(i) $\operatorname{Ass}_R R/I \subseteq \operatorname{Ass}_R R$.
(ii) *There is an exact sequence $0 \to R/I \to \bigoplus_{i=1}^n R$ for some positive integer n.*
(iii) *There is an ideal J of R such that $I = 0 :_R J$.*
(iv) $I = 0 :_R (0 :_R I)$.

Proof. (i) $\Leftrightarrow$ (ii) The assertion follows from Proposition 2.1.

(ii) $\to$ (iii) Let $f : R/I \to \bigoplus_{i=1}^n R$ be a monomorphism such that $f(1 + I) = (a_1, \ldots, a_n)$. Set $J := Ra_1 + \cdots + Ra_n$. It is easy to see that $I = 0 :_R J$.

(iii) $\Leftrightarrow$ (iv) Let $I = 0 :_R J$. Then $J \subseteq 0 :_R I$ and so $I = 0 :_R J \supseteq 0 :_R (0 :_R I) \supseteq I$ that implies $I = 0 :_R (0 :_R I)$.

(iv) $\to$ (ii) Let $0 :_R I = Ra_1 + \cdots + Ra_n$. We define $f : R/I \to \bigoplus_{i=1}^n R$ with $f(r + I) = (ra_1, \ldots, ra_n)$. Then f is a monomorphism. □

The following lemma will be quite useful in the proof of the first main theorem.

Lemma 2.3. *Let $(R, \mathfrak{m})$ be a local Noetherian ring and M a finitely generated R-module. Let $\mathfrak{p}$ be a prime ideal of R such that $\dim R/\mathfrak{p} = 1$ and $t \geqslant 1$ be an integer. Then the following conditions are equivalent*: