Journal of Algebra 360 (2012) 87–91

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Quasi-projective modules over prime hereditary noetherian V-rings are projective or injective

Hai Q. Dinh [a], Christopher J. Holston [b], Dinh V. Huynh [b,*]

[a] Department of Mathematical Sciences, Kent State University, Warren, OH, USA
[b] Department of Mathematics, Ohio University, Athens, OH, USA

ARTICLE INFO

*Article history:*
Received 30 March 2012
Available online 12 April 2012
Communicated by Louis Rowen

*MSC:*
16U10
16D40
16P40

*Keywords:*
Hereditary noetherian ring
$R$-projective module
PCI-domain
V-domain
SI-ring
Nonsingular module
Singular module

ABSTRACT

Let $\mathbb{Q}$ be the field of rational numbers. As a module over the ring $\mathbb{Z}$ of integers, $\mathbb{Q}$ is $\mathbb{Z}$-projective, but $\mathbb{Q}_{\mathbb{Z}}$ is not a projective module. Contrary to this situation, we show that over a prime right noetherian right hereditary right V-ring $R$, a right module $P$ is projective if and only if $P$ is $R$-projective. As a consequence of this we obtain the result stated in the title. Furthermore, we apply this to affirmatively answer a question that was left open in a recent work of Holston, López-Permouth and Orhan Ertag (2012) [9] by showing that over a right noetherian prime right SI-ring, quasi-projective right modules are projective or semisimple.



## 1. Introduction

We consider associative rings with identity. All modules are unitary modules. Let $M$, $N$ be right $R$-modules. The module $M$ is called $N$-projective if for each exact sequence

$$0 \to H \to N \xrightarrow{g} K \to 0$$

* Corresponding author.
*E-mail addresses:* hdinh@kent.edu (H.Q. Dinh), holston@ohio.edu (C.J. Holston), huynh@ohio.edu (D.V. Huynh).

0021-8693/$ – see front matter 
http://dx.doi.org/10.1016/j.jalgebra.2012.04.002

in Mod-$R$, and any homomorphism $f : M \to K$ there is a homomorphism $f' : M \to N$ such that $gf' = f$.

A right $R$-module $P$ is defined to be a projective module if $P$ is $N$-projective for any $N \in$ Mod-$R$. A ring $R$ is right (left) hereditary if every right (left) ideal of $R$ is projective as a right (left) $R$-module. A right and left hereditary (noetherian) ring is simply called a hereditary (noetherian) ring.

For basic properties of (quasi-)projective modules as well as concepts of modules and rings not defined here we refer to [1,7,11,12,14].

Unlike the injectivity of modules, an $R$-projective module may not be projective, in general. As an example for that, we consider the ring $\mathbb{Q}$ of rational numbers as a module over the ring $\mathbb{Z}$ of integers (cf. [1, 10(1), p. 190]). Then, since every nonzero homomorphic image of $\mathbb{Q}_{\mathbb{Z}}$ is infinite (and injective), there is no nonzero homomorphism from $\mathbb{Q}_{\mathbb{Z}}$ to $\mathbb{Z}/A$ for any ideal $A \subset \mathbb{Z}$. Hence $\mathbb{Q}_{\mathbb{Z}}$ is $\mathbb{Z}$-projective. But, obviously, $\mathbb{Q}_{\mathbb{Z}}$ is not projective. Note that $\mathbb{Z}$ is a noetherian hereditary domain, it is even a commutative PID.

Motivated by this we ask a question:

*For which noetherian hereditary domains $D$, $D$-projectivity implies projectivity?*

In this note we answer this question affirmatively for prime right noetherian right hereditary right V-rings (Corollary 4). Using this we show that the class (iii) of [9, Theorem 3.11] is not empty. This is what the authors of [9] wanted to see.

Note that a ring $R$ is called a right V-ring (after Villamyor) if every simple right $R$-module is injective. For basic properties of V-rings we refer to [13].

## 2. Results

A submodule $E$ of a module $M$ is called an essential submodule if for any nonzero submodule $A \subseteq M$, $E \cap A \neq 0$. A nonzero submodule $U \subseteq M$ is called uniform if every nonzero submodule of $U$ is essential in $U$.

A right $R$-module $N$ is called nonsingular if for any nonzero element $x \in N$ the annihilator $ann_R(x)$ of $x$ in $R$ is not an essential right ideal of $R$. A right $R$-module $S$ is called a singular module if the annihilator in $R$ of each nonzero element of $S$ is an essential right ideal of $R$. Every $R$-module $M$ has a maximal singular submodule $Z(M)$ which contains all singular submodules of $M$. This is a fully invariant submodule of $M$ and it is called the singular submodule of $M$. Clearly, $M$ is nonsingular if and only if $Z(M) = 0$. For a ring $R$, if $Z(R_R) = 0$ (resp., $Z({}_R R) = 0$) then $R$ is called right (left) nonsingular. (See, e.g., [8, p. 5].) To indicate that $M$ is a right (left) module over $R$ we write $M_R$ (resp., ${}_R M$).

**Lemma 1.** *Let $R$ be a right nonsingular right V-ring. Any nonzero $R$-projective right $R$-module $M$ is nonsingular.*

**Proof.** Assume on the contrary that $M$ contains a nonzero singular submodule $T$. As $R$ is a right V-ring, $T$ contains a maximal submodule $V$ for which we have $M/V = (T/V) \oplus (L/V)$ for some submodule $L$ of $M$ with $V \subset L$. On the other hand, there exists a maximal right ideal $B \subset R$ such that $R/B \cong T/V$ as right $R$-modules. This means there is a homomorphism $f : M \to R/B$ with $\mathrm{Ker}(f) = L$. By the definition of the $R$-projectivity, there exists a homomorphism $f' : M \to R_R$ such that $gf' = f$ where $g$ is the canonical homomorphism $R \to R/B$. However, this is impossible, because as $R$ is right nonsingular, the kernel of $f'$ must contain the singular submodule $T$ which implies $\mathrm{Ker}(gf') \neq \mathrm{Ker}(f)$. Thus $M$ does not contain a nonzero singular submodule, proving that $M$ is nonsingular. □

We would like to remark that Lemma 1 does not hold if the ring $R$ is not a right V-ring. As an example for this, we again take the ring $\mathbb{Z}$. Let $p \in \mathbb{Z}$ be a prime number, then the $p$-Prüfer group $C(p^\infty)$ is $\mathbb{Z}$-projective (cf. [1, 10(1), p. 190]), but $C(p^\infty)$ is a singular $\mathbb{Z}$-module.

If a module $M$ has finite uniform dimension, we denote its dimension by u-dim$(M)$ and call $M$ a finite dimensional module.