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Further solvable analogues of the Baer–Suzuki theorem and generation of nonsolvable groups

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ABSTRACT

Let G be an almost simple group. We prove that if $x \in G$ has prime order $p \geqslant 5$, then there exists an involution y such that $\langle x, y\rangle$ is not solvable. Also, if x is an involution then there exist three conjugates of x that generate a nonsolvable group, unless x belongs to a short list of exceptions, which are described explicitly. We also prove that if x has order 6 or 9, then there exist two conjugates that generate a nonsolvable group.

1. Introduction

The following theorem is proved in [Gue10], and provides a solvable analogue of the classical Baer–Suzuki theorem for elements of certain orders.

Theorem 1.1. *Let G be a finite group and suppose that x is an element of prime order p where $p \geqslant 5$. Then x is contained in the solvable radical of G if and only if $\langle x, x^g\rangle$ is solvable for all $g \in G$. In other words, if x is not contained in the solvable radical of G then there exists $g \in G$ such that $\langle x, x^g\rangle$ is not solvable.*

The proof of Theorem 1.1 is by induction, and it is shown that a minimal counterexample to Theorem 1.1 would have to be an almost simple group. Theorem 1.1 is then proved (in [Gue10]) with the following result for almost simple groups.

Theorem 1.2. *Let G be an almost simple group with socle G_0. Let $x \in G$ have odd prime order p. Then one of the following holds.*

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(1) *There exists* $g \in G$ *such that* $\langle x, x^g \rangle$ *is not solvable*;
(2) $p = 3$ *and* x *is a long root element in a simple group of Lie type defined over* $\mathbb{F}_3$, x *is a short root element in* $G_2(3)$, *or* x *is a pseudoreflection and* $G_0 \cong \mathrm{PSU}_d(2)$.

In this paper, we prove a result that is quite similar to Theorem 1.2.

Theorem 1.3. *Suppose that the finite group* G *is not solvable and satisfies one of the following conditions*:

(1) G *is almost simple*;
(2) $\mathrm{SL}_d(q) \leqslant G \leqslant \mathrm{GL}_d(q)$ *or* $\mathrm{SU}_d(q) \leqslant G \leqslant \mathrm{GU}_d(q)$, *and if* $d = 2$ *and* q *is odd, then* $\mathrm{SL}_2(q)$ *or* $\mathrm{SU}_2(q)$ *has even index in* G;
(3) $G \cong K/Z$, *where* $Z \leqslant Z(K)$, K *is the universal version of a group of Lie type,* $K/Z(K)$ *is simple and* $G \ncong \mathrm{SL}_2(q)$ $(q$ *odd*$)$.

If $x \in G$ *has prime order* $p \geqslant 5$ *in* $G/Z(G)$, *then there exists an involution* $y \in G$ *such that* $\langle x, y \rangle$ *is not solvable.*

Following [Ste68] (see also [GLS98, 2.2.6]), we refer to the groups K/Z in part (3) of Theorem 1.3 as finite groups of Lie type. We note that if $p \geqslant 5$ and G is almost simple, then Theorem 1.3 shows that there exists *an involution* y such that $\langle x, x^y \rangle$ is not solvable. For $\langle x, x^y \rangle$ has index 1 or 2 in $\langle x, y \rangle$ and so either both groups are solvable, or both of them are not solvable. Also, Theorem 1.2 shows that when the order of x has a prime divisor $p \geqslant 5$ and G is almost simple, there exist two conjugates that generate a nonsolvable group. In this paper we prove an analogous result for elements of order divisible by 3.

Theorem 1.4. *Suppose that* G *is an almost simple group and that* x *has order* 6 *or* 9. *Then there exists an element* $g \in G$ *such that* $\langle x, x^g \rangle$ *is not solvable.*

Theorems 1.2 and 1.4 yield the following corollary immediately.

Corollary 1.5. *Let* G *be an almost simple group with socle* G_0 *and suppose that* x *in* G *is not a* 2*-element. Then there exists* g *in* G *such that* $\langle x, x^g \rangle$ *is not solvable or* x *has order* 3 *and* x *is a long root element in a simple group of Lie type defined over* $\mathbb{F}_3$, *a pseudoreflection in* $\mathrm{PGU}_d(2)$ *or a short root element in* $G_2(3)$. *Moreover, there exist three conjugates of* x *that generate a nonsolvable group unless* $G_0 \cong \mathrm{PSU}_d(2)$ *or* $\mathrm{PSp}_d(3)$.

Guralnick, Flavell, and the author prove in [FGG10] that for all nontrivial elements x in a finite (or linear) group G, x is contained in the solvable radical of G if and only if any four conjugates of x generate a solvable group. In particular, if x is contained in an almost simple group G, then there exist four conjugates of x that generate a nonsolvable group (this result and Theorem 1.1 are obtained independently by Gordeev, Grunwald, Kunyavski, and Plotkin in [GGKP10]). Thus, if we allow x to be a 2-element, then a similar result to Corollary 1.5 is true but with four conjugates of x. Corollary 1.5 and Theorem 1.6 show that in most cases, there exist three conjugates of x that generate a nonsolvable group.

Theorem 1.6. *Let* G *be an almost simple group with socle* G_0 *and* x *an involution in* G. *Then either there exist* $g_1, g_2 \in G$ *such that* $\langle x, x^{g_1}, x^{g_2} \rangle$ *is not solvable or* (x, G_0) *belongs to Table* 1.

We note that if x is an involution, then $\langle x, x^g \rangle$ is dihedral and so we need at least three conjugate involutions to generate a nonsolvable group.

In a future work, the author hopes to improve Corollary 1.5 to find the minimal number of conjugates in an almost simple group required to generate a nonsolvable group for 2-elements as well. This requires a proof that for an element of order 4, there exist two conjugates that generate a nonsolvable group with a short list of exceptions, and that two conjugates always suffice for an element of order 8.