Journal of Algebra 359 (2012) 49–68

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A quantum analogue of the dihedral action on Grassmannians

Justin M. Allman [a], Jan E. Grabowski [b,*]

[a] Department of Mathematics, the University of North Carolina at Chapel Hill, CB #3250, Phillips Hall, Chapel Hill, NC 27599, USA
[b] Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YF, United Kingdom

ARTICLE INFO

Article history:
Received 22 November 2011
Available online 28 March 2012
Communicated by J.T. Stafford

Keywords:
Quantum Grassmannian
Twisting
Dihedral group

ABSTRACT

In recent work, Launois and Lenagan have shown how to construct a cocycle twisting of the quantum Grassmannian and an isomorphism of the twisted and untwisted algebras that sends a given quantum minor to the minor whose index set is permuted according to the n-cycle $c = (12 \cdots n)$, up to a power of q. This twisting is needed because c does not naturally induce an automorphism of the quantum Grassmannian, as it does classically and semi-classically.

We extend this construction to give a quantum analogue of the action on the Grassmannian of the dihedral subgroup of S_n generated by c and w_0, the longest element, and this analogue takes the form of a groupoid. We show that there is an induced action of this subgroup on the torus-invariant prime ideals of the quantum Grassmannian and also show that this subgroup acts on the totally nonnegative and totally positive Grassmannians. Then we see that this dihedral subgroup action exists classically, semi-classically (by Poisson automorphisms and anti-automorphisms, a result of Yakimov) and in the quantum and nonnegative settings.

1. Introduction

The Grassmannian $G(m,n)$ of m-dimensional subspaces of an n-dimensional vector space is an important geometric and algebraic object that occurs in many different contexts. It is a projective variety and its geometric structure is now well understood, including in particular a cell decomposition. It is also well known that the Grassmannian admits an action of the symmetric group S_n.

* Corresponding author.
E-mail addresses: jallman@email.unc.edu (J.M. Allman), j.grabowski@lancaster.ac.uk (J.E. Grabowski).

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.03.016

Each point in the Grassmannian may be specified by an $m \times n$ matrix of rank m and the symmetric group action is by permutation of the columns of this matrix. This action of course extends to the coordinate ring of the Grassmannian, $\mathcal{O}(G(m,n))$.

Unfortunately, several other important relations of the Grassmannian do not admit a symmetric group action. Firstly, the (real) totally nonnegative Grassmannian $G^{\mathrm{tnn}}(m,n)$ does not. Recall that a real $m \times n$ matrix is totally nonnegative if all of its $m \times m$ minors are nonnegative. The totally nonnegative Grassmannian is the space of all of these modulo the action of $\mathrm{GL}^+(m)$, the group of real $m \times m$ matrices with positive determinant. Clearly the property of being a point in $G^{\mathrm{tnn}}(m,n)$ is not preserved under arbitrary column permutation. Secondly, in recent work Launois and Lenagan [12] showed that the natural action of the Coxeter element $c = (12 \cdots n)$ on indexing sets of quantum minors does not induce an automorphism of the quantum Grassmannian $\mathcal{O}_q(G(m,n))$.

Not all is lost, however. Postnikov [17, Remark 3.3] has observed that one may define an action of c on the totally nonnegative Grassmannian by column permutation and a suitable sign correction. Yakimov [21, Theorem 0.1] has shown that c induces a Poisson automorphism of the complex Grassmannian $G(m,n)_{\mathbb{C}}$, taken with its standard Poisson structure. Indeed, it has long been known that the longest element w_0 of the symmetric group induces a Poisson anti-automorphism of $G(m,n)_{\mathbb{C}}$ and that $G(m,n)_{\mathbb{C}}$ admits an action of the maximal torus T_n of diagonal matrices. Combining these, Yakimov notes that the group $I_2(n) \ltimes T_n$ therefore acts on $G(m,n)_{\mathbb{C}}$ by Poisson automorphisms and anti-automorphisms, where $I_2(n) = \langle c, w_0 \rangle \cong D_{2n}$ is the dihedral group of order $2n$, given its Coxeter group name.

The paper [12] of Launois and Lenagan explains how one may construct a replacement for the cycling automorphism in the quantum setting by twisting the quantum Grassmannian. In this work, we show how to extend this to a quantum analogue of the above $(I_2(n) \ltimes T_n)$-action and deduce that one obtains a dihedral action on the set of torus-invariant prime ideals of $\mathcal{O}_q(G(m,n))$. We also note that one may define an $I_2(n)$-action on both the totally nonnegative and totally positive Grassmannians, though these do not admit the torus action.

Further reasons for interest in a dihedral action on the quantum Grassmannian come from the study of quasi-commuting sets of quantum minors and cluster algebras. Leclerc and Zelevinsky [14] have shown that two quantum minors in $\mathcal{O}_q(G(m,n))$ quasi-commute if and only if the column sets defining them satisfy a combinatorial condition called weak separability. As noted by Scott [19, Proposition 4], the natural action of the group $I_2(n)$ on m-subsets of $\{1,\ldots,n\}$ preserves weak separability and so the question of an analogue of the dihedral action on $\mathcal{O}_q(G(m,n))$ naturally arises here too.

In [6], the second author and Launois have observed the quantum cycling of [12] playing a role in a quantum cluster algebra structure on $\mathcal{O}_q(G(3,m))$ for $m = 6, 7, 8$ and also hints of a quantum dihedral action. Since quantum clusters are by definition quasi-commuting sets, this is not so surprising, although it should be noted that in the cases mentioned not all quantum cluster variables are quantum minors but the dihedral action is still evident. Also, Assem, Schiffler and Shramchenko [1] have studied automorphism groups of (unquantized) cluster algebras and shown that a cluster algebra of type A_{n-3} has a dihedral cluster automorphism group of order $2n$. Fomin and Zelevinsky [3] showed that $\mathcal{O}(G(2,n))$ is a cluster algebra of type A_{n-3} and hence $\mathcal{O}(G(2,n))$ has cluster automorphism group isomorphic to $I_2(n)$. It is expected that the results presented here will aid the understanding of the quantum cluster algebra structures conjectured to exist for all quantum Grassmannians.

As noted above, in the commutative setting the Grassmannian admits a symmetric group action and we are clearly a considerable distance from having a quantum analogue of this, if indeed one exists. A direction for future work would be to try to extend our dihedral action further. It would also be interesting to know whether other related geometric results can be similarly improved, such as whether the action of c on the Lusztig strata of the Grassmannian [8,21] extends to a dihedral action on the strata. We do not address these questions here, though.

The main result of this paper, the quantum analogue of the dihedral action on $\mathcal{O}_q(G(m,n))$, can be expressed in categorical language. A groupoid is a category in which every morphism has an inverse and the natural way to regard an algebra and its automorphism group is as a groupoid with one object; the latter is usually simply called a group, of course. Then what we see is that under quantisation the subgroup $I_2(n)$ of $\mathrm{Aut}(\mathcal{O}(G(m,n)))$ is replaced by a groupoid with infinitely many objects, the twists of $\mathcal{O}_q(G(m,n))$, and arrows generated by the isomorphisms we establish. This