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The chain lemma for biquaternion algebras

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ABSTRACT

Any two decompositions of a biquaternion algebra over a field F into a sum of two quaternion algebras can be connected by a chain of decompositions such that any two neighboring decompositions are $(a,b)+(c,d)$ and $(ac,b)+(c,bd)$ for some $a,b,c,d \in F^*$. A similar result is established for decompositions of a biquaternion algebra into a sum of three quaternions if F has no cubic extension.

Let A be a biquaternion algebra (i.e. a tensor product of two quaternion algebras) over a field F of characteristic different from 2. A decomposition of A into a tensor product of two quaternion algebras is not unique, and there is no canonical one. However, it turns out that any two decompositions of A can be connected by a chain of decompositions in which neighboring ones do not differ "too much". In fact in this note we prove an analogue of the chain lemma (see, for instance [L], where it is called "Common Slot Theorem") for a quaternion algebra.

So let $A = D_1 + D'_1 = D_2 + D'_2$ be two decompositions of A into a sum of two quaternion algebras (the signs $=$ and $+$ will always mean equality and addition in the Brauer group of F). Dimension count shows that this means

$$A \simeq D_1 \otimes_F D'_1 \simeq D_2 \otimes_F D'_2.$$

We call these decompositions equal if $D_1 = D_2$ and $D'_1 = D'_2$, and simply-equivalent if there exist elements $x, y, a, c \in F$ such that $D_{1\,F(\sqrt{a})} = D'_{1\,F(\sqrt{c})} = 0$ and

$$D_2 = D_1 + \left(a, x^2 - acy^2\right), \qquad D'_2 = D'_1 + \left(c, x^2 - acy^2\right). \tag{$*$}$$

Notice that, since $(ac, x^2 - acy^2) = 0$, we have $D_1 + D'_1 = D_2 + D'_2$ as soon as the equalities $(*)$ hold. We say that two decompositions of A are equivalent if they can be connected by a chain of

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decompositions in such a way that every two neighboring decompositions in this chain are simply-equivalent. The following result justifies this definition.

Proposition 1. *Any two biquaternion decompositions of A are equivalent to one another, and can be connected by a chain of length* 3. *Moreover, this bound is strict, i.e. in general two decompositions of A cannot be connected by a chain of length* 2.

Proof. Let $A=(a_1,b_1)+(c_1,d_1)=(a_2,b_2)+(c_2,d_2)$ be two decompositions of A. Assume first that the algebras (a_1,b_1) and (a_2,b_2) have a common splitting quadratic extension. In this case we may suppose that $a_1=a_2$. Hence $(c_1,d_1)+(c_2,d_2)=(a_1,b_1b_2)$, so (c_1,d_1) and (c_2,d_2) have a common splitting quadratic extension [A]. Therefore, we may suppose that $c_1=c_2$. This implies that $(a_1,b_1b_2)=(c_1,d_1d_2)$. Denote this algebra by Q. We have $Q_{F(\sqrt{a_1})}=Q_{F(\sqrt{c_1})}=0$. It is easy to verify that $Q\simeq(a_1,x^2-a_1c_1y^2)$ for some $x,y\in F$. Hence

$$(a_2,b_2)=(a_1,b_1)+(a_1,b_1b_2)=(a_1,b_1)+\big(a_1,x^2-a_1c_1y^2\big),$$

and

$$(c_2,d_2)=(c_1,d_1)+(c_1,d_1d_2)=(c_1,d_1)+\big(a_1,x^2-a_1c_1y^2\big).$$

In particular, the decompositions $(a_1,b_1)+(c_1,d_1)$ and $(a_2,b_2)+(c_2,d_2)$ are simply-equivalent. This implies that in the general case it suffices to find $x_1,y_1,x_2,y_2\in F$ such that the algebras $(a_1,b_1(x_1^2-a_1c_1y_1^2))$ and $(a_2,b_2(x_2^2-a_2c_2y_2^2))$ have a common quadratic splitting extension. This certainly will be the case if the form

$$\big\langle a_1,b_1\big(x_1^2-a_1c_1y_1^2\big),-a_2,-b_2\big(x_2^2-a_2c_2y_2^2\big)\big\rangle$$

is isotropic. Notice that we can modify c_1 and c_2 to any values of the forms $\langle c_1,d_1,-c_1d_1\rangle$ and $\langle c_2,d_2,-c_2d_2\rangle$ respectively. Thus it suffices to show that the form

$$\langle a_1,b_1\rangle\perp -a_1b_1\langle c_1,d_1,-c_1d_1\rangle\perp\langle -a_2,-b_2\rangle\perp a_2b_2\langle c_2,d_2,-c_2d_2\rangle$$

is isotropic. But the last form is 10-dimensional, belongs to $I^2(F)$ and its Clifford invariant is equal to $(a_1,b_1)+(c_1,d_1)+(a_2,b_2)+(c_2,d_2)=0$. In particular, this form belongs to $I^3(F)$ [P]. Since any 10-dimensional form from $I^3(F)$ is isotropic [P], we are done.

An example of two decompositions which cannot be connected by a chain of length 2 is as follows. Let k be a field, $a,b,c\in k^*$, $\langle\langle a,b,c\rangle\rangle\neq 0$, $(a,b)_{k(\sqrt{c})}\neq 0$, $F=k((t))$, $A=(a,b)+(c,t)=(c,t)+(a,b)$. Suppose that these decompositions are connected by a chain of length at most 2. Then the index of $(a,b)+(c,t)+(c',x^2-a'c'y^2)$ is at most 2 for some $x,y\in F$, $a'\in D(\langle a,b,-ab\rangle)$, $c'\in D(\langle c,t,-ct\rangle)$, where, as usual, by $D(\varphi)$ we denote the set of nonzero values of the quadratic form φ. Obviously, we may assume that c' equals either c, or t, or $-ct$. We will consider these cases one by one.

(i) Assume $c'=c$. The condition $(a,b)_{k(\sqrt{c})}\neq 0$ is equivalent to the form $\langle a,b,-ab,-c\rangle$ being anisotropic. Suppose $x,y\in F$, and either $x\neq 0$, or $y\neq 0$. Then $x^2-a'cy^2\in k^*F^{*2}$, hence $(a,b)+(c,t)+(c,x^2-a'cy^2)=(a,b)+(c,et)$ for some $e\in k^*$. Since $(a,b)_{k(\sqrt{c})}\neq 0$, and $c\notin k^{*2}$ (for $\langle\langle a,b,c\rangle\rangle\neq 0$), we get by Prop. 2.4 in [T] that $\operatorname{ind}(a,b)\otimes(c,et)=4$, a contradiction.

(ii) Assume $c'=t$. Obviously, $x^2-a'ty^2\in F^{*2}\cup -a'tF^{*2}$, hence $(a,b)+(c,t)+(t,x^2-a'ty^2)$ equals either $(a,b)+(c,t)$, or $(a,b)+(a'c,t)$. If the index of the last algebra is 2, then again by Prop. 2.4 of [T] either $a'c\in D(\langle a,b,-ab\rangle)$, or $a'c\in k^{*2}$, which implies that $c\in D(\langle\langle a,b\rangle\rangle)$, a contradiction in view of the hypothesis $\langle\langle a,b,c\rangle\rangle\neq 0$.

(iii) The case $c'=-ct$ is quite similar to case (ii). The algebra $(a,b)+(c,t)+(-ct,x^2+a'cty^2)$ equals either $(a,b)+(c,t)$, or $(a,b)+(c,t)+(-ct,a')=(a,b)+(-c,a')+(a'c,t)$. If the index of the last algebra is 2, then as in case (ii) $a'c\in D(\langle a,b,-ab\rangle)$, or $a'c\in k^{*2}$, which is impossible. □