Journal of Algebra 349 (2012) 353–363

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Incompressibility of orthogonal Grassmannians of rank 2

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ARTICLE INFO

Article history:
Received 23 July 2011
Available online 10 October 2011
Communicated by Eva Bayer-Fluckiger

Keywords:
Canonical dimension
Incompressibility
Orthogonal Grassmannians
Motivic decomposition

ABSTRACT

For a nondegenerate quadratic form φ on a vector space V of dimension $2n+1$, let X_d be the variety of d-dimensional totally isotropic subspaces of V. We give a sufficient condition for X_2 to be 2-incompressible, generalizing in a natural way the known sufficient conditions for X_1 and X_n. Key ingredients in the proof include the Chernousov–Merkurjev method of motivic decomposition as well as Pragacz and Ratajski's characterization of the Chow ring of $(X_2)_E$, where E is a field extension splitting φ.

1. Preliminaries

Before stating our main theorem (2.3) below, we recall the notions of canonical p-dimension, p-incompressibility, and higher Witt index.

Let X be a scheme over a field F, and let p be a prime or zero. A field extension K of F is called a *splitting field of* X (or is said to *split* X) if $X(K) \neq \emptyset$. A splitting field K is called *p-generic* if, for any splitting field L of X, there is an F-place $K \rightharpoonup L'$ for some finite extension L'/L of degree prime to p. In particular, K is 0-generic if for any splitting field L there is an F-place $K \rightharpoonup L$.

The canonical p-dimension of a scheme X over F was originally defined [1,2] as the minimal transcendence degree of a p-generic splitting field K of X. When X is a smooth complete variety, the original algebraic definition is equivalent to the following geometric one [2,3].

Definition 1.1. Let X be a smooth complete variety over F. The *canonical p-dimension* $\operatorname{cdim}_p(X)$ *of* X is the minimal dimension of the image of a morphism $X' \to X$, where X' is a variety over F admitting a dominant morphism $X' \to X$ with $F(X')/F(X)$ finite of degree prime to p. The canonical 0-dimension of X is thus the minimal dimension of the image of a rational morphism $X \dashrightarrow X$.

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.08.032

In the case $p = 0$, we will drop the p and speak simply of *generic* splitting fields and canonical *dimension* $\mathrm{cdim}(X)$.

For a third definition of canonical p-dimension as the essential p-dimension of the detection functor of a scheme X, we refer the reader to Merkurjev's comprehensive exposition [3] of essential dimension.

For a smooth complete variety X, the inequalities

$$\mathrm{cdim}_p(X) \leqslant \mathrm{cdim}(X) \leqslant \dim(X)$$

are clear from Definition 1.1. Note also that if X has a rational point, then $\mathrm{cdim}(X) = 0$ (though the converse is not true).

Definition 1.2. When a smooth complete variety X has canonical p-dimension as large as possible, namely $\mathrm{cdim}_p(X) = \dim(X)$, we say that X is *p-incompressible*.

It follows immediately that if X is p-incompressible, it is also *incompressible* (i.e. 0-incompressible).

We next recall the definitions of absolute and relative higher Witt indices, introduced by Knebusch in [4]. Our discussion follows [5, §90]. The *Witt index* $\mathfrak{i}_0(\varphi)$ of a quadratic form φ is the number of copies of the hyperbolic plane $\mathbb{H}$ which appear in the Witt decomposition of φ. Now let φ be a nondegenerate quadratic form over a field F and set $F_0 := F$ and $\varphi_0 := \varphi_{an}$, the anisotropic part of φ. We proceed to recursively define $F_k := F_{k-1}(\varphi_{k-1})$, $\varphi_k := (\varphi_{F_k})_{an}$ for $k = 1, 2, \ldots,$ stopping at F_h such that $\dim \varphi_h \leqslant 1$.

Definition 1.3. For $k \in \{0, 1, \ldots, h\}$, the *$k$-th absolute higher Witt index* $\mathfrak{j}_k(\varphi)$ *of* φ is defined to be $\mathfrak{i}_0(\varphi_{F_k})$. For $k \in \{1, 2, \ldots, h\}$, the *$k$-th relative higher Witt index* $\mathfrak{i}_k(\varphi)$ *of* φ is defined to be the difference

$$\mathfrak{i}_k(\varphi) := \mathfrak{j}_k(\varphi) - \mathfrak{j}_{k-1}(\varphi).$$

The 0*-th relative higher Witt index of* φ is the usual Witt index $\mathfrak{i}_0(\varphi)$.

It follows from the definition that

$$0 \leqslant \mathfrak{j}_0(\varphi) < \mathfrak{j}_1(\varphi) < \cdots < \mathfrak{j}_h(\varphi) = \big[(\dim \varphi)/2\big].$$

Moreover, it can be shown that the set $\{\mathfrak{j}_0(\varphi), \ldots, \mathfrak{j}_h(\varphi)\}$ of absolute higher Witt indices of φ is equal to the set of all Witt indices $\mathfrak{i}_0(\varphi_K)$ for K an extension field of F.

2. Introduction

Let φ be a nondegenerate quadratic form on a vector space V of dimension $2n + 1$ over a field F. Associated to φ there are smooth projective varieties $X_1, X_2, \ldots, X_n$, where X_d is the variety of d-dimensional totally isotropic subspaces of V. The variety X_1 is simply the projective quadric hypersurface associated to the quadratic form φ.

We recall the following result proved in [6] and also in [5, Ch. XIV and §90].

Theorem 2.1 *(Karpenko, Merkurjev). If the quadric X_1 is anisotropic, then*

$$\mathrm{cdim}_2(X_1) = \mathrm{cdim}(X_1) = \dim(X_1) - \mathfrak{i}_1(\varphi) + 1.$$

In particular, X_1 is 2-incompressible if and only if $\mathfrak{i}_1(\varphi) = 1$.