Journal of Algebra 347 (2011) 224–246

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Approximately norm-unital products on C^*-algebras, and a non-associative Gelfand–Naimark theorem

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ARTICLE INFO

Article history:
Received 21 June 2011
Available online 12 July 2011
Communicated by Efim Zelmanov

MSC:
46B04
46L05
46L70

Keywords:
C^*-algebra
JB^*-triple
Non-associative C^*-algebra
Automatic w^*-continuity

ABSTRACT

We describe the non-associative products on a C^*-algebra A which convert the Banach space of A into a Banach algebra having an approximate unit bounded by 1, and determine among them those which are associative. As a consequence, if such a product p satisfies $p(a,b)^\square = p(b^\square, a^\square)$ and $\|p(a^\square, a)\| = \|a\|^2$, for all $a, b \in A$ and some conjugate-linear vector space involution $\square$ on A, then p is associative. The proof of the above result involves also a new Gelfand–Naimark type theorem asserting that non-associative C^*-algebras (defined verbatim as in the associative case, but removing associativity) are alternative if and only if they have an approximate unit bounded by 1.

1. Introduction

The necessity of considering different products on a common Banach space originated in Arens' early paper [4], where it is shown that, given a Banach algebra A, there are two quite natural ways to extend the product of A to the bidual A^{**} of A, that the two products on A^{**} obtained in these ways need not coincide, and that none of them is better than the other. Moreover, the coincidence of the two Arens products on A^{**} is equivalent to the existence of a separately w^*-continuous bilinear extension to A^{**} of the product of A. In this case, the common value of the two Arens products becomes the unique separately w^*-continuous bilinear extension to A^{**} of the product of A, and the Banach algebra A is called Arens-regular. Consequently, since C^*-algebras are Arens-regular, and the bidual of a C^*-algebra is a von Neumann algebra in a natural way, several authors have been

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[1] Partially supported by Junta de Andalucía grants FQM 0199 and FQM 3737, and Projects I+D MCYT-2006-15546-C02-02 and MTM-2007-65959.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.06.020

interested in the question of the "automatic w^*-continuity" of operators or products on von Neumann algebras and related mathematical models. The starting point in this line of work could be dated in the Godefroy and Iochum paper [18], whereas, as a relevant recent representative, we could cite the Blecher and Magajna paper [8].

A nice sample of an automatic w^*-continuity theorem, of special interest in relation to the present paper, is the following (see [18, Theorem II.1] and its proof).

Theorem 1.1. *Let A be a von Neumann algebra, and let $p : A \times A \to A$ be a bilinear mapping satisfying*

(α) $\|p(a,b)\| \leqslant \|a\|\|b\|$ *for all* $a, b \in A$,
(β) $p(a, \mathbf{1}) = p(\mathbf{1}, a) = a$ *for every* $a \in A$, *where* $\mathbf{1}$ *stands for the unit of* A.

Then p is separately w^*-continuous.

We proved in [38, Theorem 5.17] that, if A is the von Neumann algebra of all bounded linear operators on a complex Hilbert space, and if p is as in Theorem 1.1, then there exists a real number $\alpha \in [0, 1]$ such that

$$p(a,b) = \alpha ab + (1-\alpha)ba$$

for all $a, b \in A$. Based on this result, we proved later that, if A and p are as in Theorem 1.1, then there is a central self-adjoint element $\psi \in A$, with $0 \leqslant \psi \leqslant \mathbf{1}$, such that

$$p(a,b) = \psi ab + (\mathbf{1} - \psi)ba$$

for all $a, b \in A$. Indeed, this last result, collected in Corollary 2.8 of the present paper, follows straightforwardly from [39, Lemma 1.1] and the implication $(i) \Rightarrow (vi)$ in [39, Corollary 2.6]. We note that the result just reviewed contains Theorem 1.1, and that its proof in [39] does not involve the conclusion in that theorem.

We begin the present paper by showing that, if A and p are as in Theorem 1.1, but if we relax the requirement (β) in that theorem to the one that there exists a norm-one element $u \in A$ such that $p(a,u) = p(u,a) = a$ for every $a \in A$, or even to the one that

(γ) there is a net a_λ in the closed unit ball of A such that $\lim p(a, a_\lambda) = \lim p(a_\lambda, a) = a$ for every $a \in A$,

then p can be also reasonably described (see Corollary 2.6), so that the separate w^*-continuity of p follows straightforwardly. Even, if we relax in addition the requirement that A is a von Neumann algebra to the one that A is merely a C^*-algebra, then a (rather more involved) description of the product p can be given (see Theorem 2.11).

In Section 3, we describe and characterize those bilinear products p, on a C^*-algebra A, which satisfy requirements (α) and (γ) above (called approximately norm-unital products throughout the paper), and which are in fact associative (see Theorem 3.5). Specializations of this result in the particular case that A is a von Neumann algebra are also discussed (see Corollaries 3.3 and 3.6). Among the information contained in Theorem 3.5, we emphasize the fact that approximately norm-unital alternative products on a C^*-algebra are associative.

In Section 4, we introduce non-associative C^*-algebras (defined verbatim as in the associative case, but removing associativity), and show that non-associative C^*-algebras are alternative if (and only if) they have an approximate unit bounded by 1 (Theorem 4.7). Since alternative C^*-algebras are well understood [20,31], the above result can be seen as a non-associative version of the Gelfand–Naimark theorem. As a consequence, approximately norm-unital non-associative C^*-products on a C^*-algebra are associative (Corollary 4.9). Section 4 contains also abundant examples of non-associative C^*-products which are not alternative (see Propositions 4.10 and 4.11, and Remark 4.12).