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Class-preserving automorphisms and inner automorphisms of certain tree products of groups

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ABSTRACT

In general, a class-preserving automorphism of generalized free products of nilpotent groups need not be an inner automorphism. We prove that every class-preserving automorphism of tree products of finitely generated nilpotent or free groups, amalgamating infinite cyclic subgroups, is inner. It follows that the outer automorphism groups of such tree products, amalgamating infinite cyclic subgroups, are residually finite.

1. Introduction

An automorphism α of a group G is called a *class-preserving* (or point-wise inner) automorphism if, for each $g \in G$, $\alpha(g)$ and g are conjugate in G. Burnside [3] constructed a group of order 3^6 admitting class-preserving automorphisms which are not inner. There are many nilpotent groups having class-preserving automorphisms which are not inner [3,12,13]. On the other hand, Grossman [5] defined that a group G has *Property A* if all class-preserving automorphisms of G are inner. She proved that free groups and fundamental groups of compact orientable surfaces have Property A. Endimioni [4] showed that free nilpotent groups have Property A. However, Segal [12] constructed a finitely

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generated torsion-free nilpotent group which does not have Property A. Since Grossman [5] proved that outer automorphism groups of finitely generated conjugacy separable groups with Property A are residually finite, those mentioned groups with Property A have residually finite outer automorphism groups. Outer automorphism groups of Fuchsian groups [1,9] and most of Seifert 3-manifold groups [2] are residually finite. Residual finiteness of outer automorphism groups of certain tree products, amalgamating central subgroups, was proved in [14].

Nontrivial free products of groups have Property A [1,10]. However, there are generalized free products of finite (or infinite) nilpotent groups which do not have Property A (Examples 4.1 and 4.10). In this paper, we prove a criterion that certain generalized free products have Property A (Theorem 2.6). Using this, we can prove that tree products of groups, amalgamating central subgroups, have Property A (Corollary 3.6). This was first proved in [14]. Our proof is quite short comparing with the proof in [14]. Moreover, we can show that tree products of finitely generated nilpotent or free groups, amalgamating infinite cyclic subgroups, have Property A (Theorem 4.8). Since these tree products are conjugacy separable [11], by Grossman [5], outer automorphism groups of tree products of finitely generated nilpotent or free groups, amalgamating infinite cyclic subgroups, are residually finite.

Throughout this paper we use standard notation and terminology.

A group G is *residually finite* ($\mathcal{RF}$) if, for each nontrivial element $x \in G$, there exists a finite homomorphic image $\overline{G}$ of G such that the image of x in $\overline{G}$ is not trivial.

A group G is *conjugacy separable* if, for each pair of elements $x, y \in G$ such that x and y are not conjugate in G, there exists a finite homomorphic image $\overline{G}$ of G such that the images of x and y in $\overline{G}$ are not conjugate in $\overline{G}$.

If A and B are groups, then $A *_H B$ denotes the generalized free product of A and B amalgamating H.

$x \sim_G y$ means that x and y are conjugate in G, otherwise $x \nsim_G y$.

We use $\operatorname{Inn} g$ to denote the inner automorphism of G induced by $g \in G$.

$\operatorname{Out}(G)$ denotes the outer automorphism group, $\operatorname{Aut}(G)/\operatorname{Inn}(G)$, of G.

$C_G(g)$ denotes the centralizer of g in G and $Z(G)$ denotes the center of G.

Definition 1.1. By a *class-preserving* (*or point-wise inner*) *automorphism* of a group G we mean an automorphism α which is such that, for each $g \in G$, there exists $k_g \in G$, depending on g, so that $\alpha(g) = k_g^{-1} g k_g$.

Definition 1.2. (See [5].) A group G has *Property A* if for each class-preserving automorphism α of G, there exists a single element $k \in G$ such that $\alpha(g) = k^{-1} g k$ for all $g \in G$, i.e., $\alpha = \operatorname{Inn} k$.

We shall use the following results:

Theorem 1.3. *(See [5, Grossman].) Let B be a finitely generated, conjugacy separable group with Property A. Then* $\operatorname{Out}(B)$ *is* $\mathcal{RF}$.

Theorem 1.4. *(See [8, Theorem 4.6].) Let $G = A *_H B$ and let $x \in G$ be of minimal length in its conjugacy class. Suppose that $y \in G$ is cyclically reduced, and that $x \sim_G y$.*

(1) *If $\|x\| = 0$, then $\|y\| \leqslant 1$ and, if $y \in A$, then there is a sequence $h_1, h_2, \ldots, h_r$ of elements in H such that $y \sim_A h_1 \sim_B h_2 \sim_A \cdots \sim_{A(B)} h_r = x$.*
(2) *If $\|x\| = 1$, then $\|y\| = 1$ and, either $x, y \in A$ and $x \sim_A y$, or $x, y \in B$ and $x \sim_B y$.*
(3) *If $\|x\| \geqslant 2$, then $\|x\| = \|y\|$ and $y \sim_H x^*$ where x^* is a cyclic permutation of x.*

2. Main results

Definition 2.1. Let G be a group and H be a subgroup of G. Then G is called H-**self conjugate** if $h \sim_G k$ for $h, k \in H$ then $h = k$.