Journal of Algebra 342 (2011) 161–174

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Fixed points of coprime operator groups

Cristina Acciarri [a], Pavel Shumyatsky [b,*]

[a] *Via Francesco Crispi n. 81, 63047 San Benedetto del Tronto (AP), Italy*
[b] *Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil*

A R T I C L E   I N F O

Article history:
Received 11 April 2011
Available online 23 June 2011
Communicated by E.I. Khukhro

Dedicated to Professor Javier Otal on the occasion of his 60th birthday

MSC:
20D45

Keywords:
Automorphisms
Centralizers
Derived groups
Associated Lie rings
Finite groups

A B S T R A C T

Let m be a positive integer and A an elementary abelian group of order q^r with $r \geqslant 2$ acting on a finite q'-group G. We show that if for some integer d such that $2^d \leqslant r-1$ the dth derived group of $C_G(a)$ has exponent dividing m for any $a \in A^{\#}$, then $G^{(d)}$ has $\{m,q,r\}$-bounded exponent and if $\gamma_{r-1}(C_G(a))$ has exponent dividing m for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ has $\{m,q,r\}$-bounded exponent.

1. Introduction

Let A be a finite group acting coprimely on a finite group G. It is well known that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has strong influence over the structure of G. To exemplify this we mention the following results.

The celebrated theorem of Thompson [18] says that if A is of prime order and $C_G(A) = 1$, then G is nilpotent. On the other hand, any nilpotent group admitting a fixed-point-free automorphism of prime order q has nilpotency class bounded by some function $h(q)$ depending on q alone. This result is due to Higman [6]. The reader can find in [8] and [9] an account on the more recent developments related to these results. The next result is a consequence of the classification of finite simple groups [21]: If A is a group of automorphisms of G whose order is coprime to that of G and $C_G(A)$ is nilpotent or has

* Corresponding author.
E-mail addresses: acciarricristina@yahoo.it (C. Acciarri), pavel@unb.br (P. Shumyatsky).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.06.013

odd order, then G is soluble. Once the group G is known to be soluble, there is a wealth of results bounding the Fitting height of G in terms of the order of A and the Fitting height of $C_G(A)$. This direction of research was started by Thompson in [19]. The proofs mostly use representation theory in the spirit of the Hall–Higman work [5]. A general discussion of these methods and their use in numerous fixed-point theorems can be found in Turull [20].

Following the solution of the restricted Burnside problem it was discovered that the exponent of $C_G(A)$ may have strong impact over the exponent of G. Remind that a group G is said to have exponent m if $x^m = 1$ for every $x \in G$ and m is the minimal positive integer with this property. The next theorem was obtained in [10].

Theorem 1.1. *Let q be a prime, m a positive integer and A an elementary abelian group of order q^2. Suppose that A acts as a coprime group of automorphisms on a finite group G and assume that $C_G(a)$ has exponent dividing m for each $a \in A^\#$. Then the exponent of G is $\{m, q\}$-bounded.*

Here and throughout the paper $A^\#$ denotes the set of nontrivial elements of A. The proof of the above result involves a number of deep ideas. In particular, Zelmanov's techniques that led to the solution of the restricted Burnside problem [24] are combined with the Lubotzky–Mann theory of powerful p-groups [13], Lazard's criterion for a pro-p group to be p-adic analytic [11], and a theorem of Bakhturin and Zaicev on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI [1].

Another quantitative result of similar nature was proved in the paper of Guralnick and the second author [4].

Theorem 1.2. *Let q be a prime, m a positive integer. Let G be a finite q'-group acted on by an elementary abelian group A of order q^3. Assume that $C_G(a)$ has derived group of exponent dividing m for each $a \in A^\#$. Then the exponent of G' is $\{m, q\}$-bounded.*

Note that the assumption that $|A| = q^3$ is essential here and the theorem fails if $|A| = q^2$. The proof of Theorem 1.2 depends on the classification of finite simple groups.

It was natural to expect that Theorems 1.1 and 1.2 admit a common generalization that would show that both theorems are part of a more general phenomenon. Let us denote by $\gamma_i(H)$ the ith term of the lower central series of a group H and by $H^{(i)}$ the ith term of the derived series of H. The following conjecture was made in [17].

Conjecture 1.3. *Let q be a prime, m a positive integer and A an elementary abelian group of order q^r with $r \geqslant 2$ acting on a finite q'-group G.*

(1) *If $\gamma_{r-1}(C_G(a))$ has exponent dividing m for any $a \in A^\#$, then $\gamma_{r-1}(G)$ has $\{m, q, r\}$-bounded exponent*;
(2) *If, for some integer d such that $2^d \leqslant r - 1$, the dth derived group of $C_G(a)$ has exponent dividing m for any $a \in A^\#$, then the dth derived group $G^{(d)}$ has $\{m, q, r\}$-bounded exponent.*

The main purpose of the present paper is to confirm Conjecture 1.3. Theorem 6.1 and Theorem 7.4 show that both parts of the conjecture are correct. The main novelty of the paper is the introduction of the concept of A-special subgroups of G (see Section 3). Using the classification of finite simple groups it is shown in Section 4 that the A-invariant Sylow p-subgroups of $G^{(d)}$ are generated by their intersections with A-special subgroups of degree d. This enables us to reduce the proof of Conjecture 1.3 to the case where G is a p-group, which can be treated via Lie methods. The idea of this kind of reduction has been anticipated already in [4]. In Section 6 we give a detailed proof of part (2) of Conjecture 1.3. In Section 7 we briefly describe how the developed techniques can be used to prove part (1) of Conjecture 1.3.

Throughout the article we use the term "$\{a, b, c, \ldots\}$-bounded" to mean "bounded from above by some function depending only on the parameters $a, b, c, \ldots$".