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# Groups with infinite mod-$p$ Schur multiplier

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ABSTRACT

The non-finiteness of the mod-$p$ Schur multiplier of a finitely generated group $G$ with trivial center implies the existence of uncountably many non-residually finite, non-isomorphic central extension groups with kernel $C_p$ (see Thm. A). This phenomenon is related to the comparison of the cohomology of the profinite completion $\hat{G}$ of a group $G$ and the cohomology of the group $G$ itself (see Thm. B); according to J-P. Serre [8] a group $G$ is *good* if the cohomologies of $\hat{G}$ and $G$ are naturally isomorphic on finite coefficients. It is shown that non-uniform arithmetic lattices of algebraic rank 1 groups over local fields of positive characteristic $p$ are not good (see Thm. C).



## 1. Introduction

For a group $G$ and a prime number $p$ the abelian group $H_2(G, \mathbb{F}_p)$ is also called the *mod-p Schur multiplier*. This group is related to the property of being finitely presented in a straight forward way, i.e., if $G$ is finitely generated and $H_2(G, \mathbb{F}_p)$ is infinite, then $G$ is not finitely presented and has infinite Schur multiplier $H_2(G, \mathbb{Z})$. In [3], R. Grigorchuk proved that the so-called first Grigorchuk group $G_1$ has infinite mod-2 Schur multiplier showing that it is also not finitely presented. A. Erschler (see [2]) used this result for constructing uncountably many non-isomorphic, non-residually finite, central extension groups of $G_1$ with kernel $C_2 = \mathbb{Z}/2\mathbb{Z}$. Our first theorem shows that this result holds for any finitely generated group with trivial center and infinite mod-$p$ Schur multiplier (see Thm. 2.4).

**Theorem A.** *Let $G$ be a finitely generated group with trivial center, i.e., $Z(G) = 1$. Then the following are equivalent.*

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(i) *$H_2(G, \mathbb{F}_p)$ is infinite.*
(ii) *There exist uncountably many central extensions*

$$1 \longrightarrow C_p \longrightarrow X \longrightarrow G \longrightarrow 1 \tag{1.1}$$

*up to equivalence or weak equivalence.*
(iii) *There exist uncountably many non-residually finite, non-isomorphic extension groups $X$ of $G$ with kernel $C_p$.*

Our second target is to study *goodness*, an important cohomological property introduced by J-P. Serre in [8, §I.2.6, Ex. (b)]. Here we call a group $G$ with profinite completion $i_G : G \to \hat{G}$ *p-good*, if for every finite left $\hat{G}$-module $M$ of $p$-power order the natural maps

$$i_G^k(M) : H^k(\hat{G}, M) \longrightarrow H^k(G, M) \tag{1.2}$$

are isomorphisms for all $k \geqslant 0$, i.e., $G$ is good if, and only if, $G$ is $p$-good for all prime numbers $p$. Our second theorem shows a quite astonishing fact: Finitely generated $p$-good groups must satisfy a strong cohomological finiteness condition (see Thm. 2.5).

**Theorem B.** *Let $G$ be a finitely generated $p$-good group. Then for every finite left $G$-module $M$ of $p$-power order and all $k \geqslant 0$ the group $H^k(G, M)$ is finite.*

As was observed in [4] the congruence subgroup property for an arithmetic group implies the failure of goodness, and the goodness was conjectured for arithmetic subgroups of $\mathrm{SL}_2(\mathbb{C})$ that fail to have the congruence subgroup property. In Remark 5.3 of [4] the authors mentioned that for arithmetic groups over a field of positive characteristic further investigation would be necessary. Our final result shows that indeed the failure of the congruence subgroup property for such arithmetic groups does not necessarily imply goodness (see Thm. 3.4).

**Theorem C.** *Let $\mathbf{k}$ be a global field of positive characteristic $p$, and let $\mathbf{k}_v$ be the completion of $\mathbf{k}$ with respect to $v$, where $v$ is a non-archimedean place of $\mathbf{k}$. Let $\mathbf{G}$ be a connected, simply-connected algebraic group defined over $\mathbf{k}$ which is absolutely almost simple of $\mathbf{k}_v$-rank 1. Let $G = \mathbf{G}(\mathbf{k}_v)$, and let $\Gamma$ be a non-uniform* arithmetic *lattice in $G$. Then $\Gamma$ is not good.*

Note that cocompact lattices $\Gamma \subseteq G = \mathbf{G}(K_v)$ of algebraic groups of this type are virtually free and therefore good.

## 2. Cohomology and finitely generated groups

### *2.1. Group extensions and extension groups*

A short exact sequence of groups

$$\mathbf{s}: \qquad 1 \longrightarrow C \longrightarrow X \longrightarrow G \longrightarrow 1 \tag{2.1}$$

is called a *group extension* of $G$ with kernel $C$. In order to avoid confusion we call the group $X$ itself an *extension group* of $G$ with kernel $C$. If $C \subseteq Z(X)$, then (2.1) is a *central extension* of $G$ with kernel $C$, and $X$ is a *central extension group* of $G$ with kernel $C$. On the class of short exact sequences (2.1) one has two equivalence relations. More precisely, two group extensions $\mathbf{s} : 1 \to C \to X \to G \to 1$ and $\mathbf{t} : 1 \to C \to Y \to G \to 1$ are called *weakly equivalent*—or for short $\mathbf{s} \approx \mathbf{t}$—if there exist isomorphisms $\alpha : C \to C$, $\beta : X \to Y$ and $\gamma : G \to G$ making the diagram