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Weyl denominator identity for affine Lie superalgebras with non-zero dual Coxeter number ☆

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ABSTRACT

Weyl denominator identity for the affinization of a simple finite-dimensional Lie superalgebra with non-zero Killing form was stated by V. Kac and M. Wakimoto and was proven by them for the defect one case. In this paper we prove this identity.

0. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra with a non-zero Killing form, which is not a Lie algebra. Their complete list is $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(m)$, $F(4)$, $G(3)$ and $D(m,n)$ with $m \neq n+1$, see [K1].

Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$. Let $\mathfrak{h}$ (resp., $\hat{\mathfrak{h}}$) be the Cartan subalgebra in $\mathfrak{g}$ (resp., in $\hat{\mathfrak{g}}$) and let $(-,-)$ be the bilinear form on $\hat{\mathfrak{h}}^*$ which is induced by the Killing form on $\mathfrak{g}$. Let Δ (resp., $\hat{\Delta}$) be the root system of $\mathfrak{g}$ (resp., of $\hat{\mathfrak{g}}$). We set

$$\Delta^{\#} := \{\alpha \in \Delta_{\bar{0}} \mid (\alpha,\alpha) > 0\}.$$

Then $\Delta^{\#}$ is a root system of a simple Lie algebra. Let $\hat{\Delta}^{\#}$ be the affinization of $\Delta^{\#}$. Denote by $\hat{W}^{\#}$ (resp., $W^{\#}$) the subgroup of $GL(\hat{\mathfrak{h}})$ generated by the reflections s_α: $\alpha \in \hat{\Delta}_{\bar{0}}$, $(\alpha,\alpha) > 0$ (resp., s_α: $\alpha \in \Delta^{\#}$). Then $W^{\#}$ is the Weyl group of $\Delta^{\#}$ and $\hat{W}^{\#}$ is the corresponding affine Weyl group. Recall that $\hat{W}^{\#} = W^{\#} \ltimes T$, where $T \subset \hat{W}^{\#}$ is the translation group, see [K3, Chapter 6]. Let Π be a set of simple roots for $\mathfrak{g}$, and let $\hat{\Pi} = \Pi \cup \{\alpha_0\}$ be the corresponding set of simple roots for $\hat{\mathfrak{g}}$. Let Δ_+,

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$\hat{\Delta}_+$ be the corresponding sets of positive roots. We set

$$R := \frac{\prod_{\alpha\in\Delta_{+,0}}(1-e^{-\alpha})}{\prod_{\alpha\in\Delta_{+,1}}(1+e^{-\alpha})}, \qquad \hat{R} := \frac{\prod_{\alpha\in\hat{\Delta}_{+,0}}(1-e^{-\alpha})}{\prod_{\alpha\in\hat{\Delta}_{+,1}}(1+e^{-\alpha})}.$$

Following [KW], we call R the *Weyl denominator* and $\hat{R}$ the *affine Weyl denominator*. The Weyl denominator identity stated by V. Kac and M. Wakimoto in [KW] is as follows

$$\hat{R}e^{\hat{\rho}} = \sum_{w\in T} w\big(Re^{\hat{\rho}}\big),$$

where $\hat{\rho}\in\hat{\mathfrak{h}}^*$ is such that $2(\hat{\rho},\alpha)=(\alpha,\alpha)$ for each $\alpha\in\Pi$. Another form of this identity is given in formula (2). In this paper we prove this identity.

In the paper [G] we proved the analog of Weyl denominator identity for finite-dimensional Lie superalgebras (also formulated and partially proven by Kac–Wakimoto); this result is used in the proof below.

1. Kac–Moody superalgebras

The notions of a Kac–Moody superalgebras and its Weyl group were introduced in [S]. We recall some definitions below and then prove Lemmas 1.3.2, 1.5.1. In the sequel, we will apply these lemmas to the case of affine Lie superalgebras; in this case the lemmas can be also verified using the explicit description of root systems.

1.1. Construction of Kac–Moody superalgebras. Let $A=(a_{ij})$ be an $n\times n$-matrix over $\mathbb{C}$ and let τ be a subset of $I:=\{1,\dots,n\}$. Let $\mathfrak{g}=\mathfrak{g}(A,\tau)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be the associated Lie superalgebra constructed as in [K2,K3]. Recall that, in order to construct $\mathfrak{g}(A,\tau)$, one considers a realization of A, i.e. a triple $(\mathfrak{h},\Pi,\Pi^\vee)$, where $\mathfrak{h}$ is a vector space of dimension $n+\operatorname{corank}A$, $\Pi\subset\mathfrak{h}^*$ (resp. $\Pi^\vee\subset\mathfrak{h}$) is a linearly independent set of vectors $\{\alpha_i\}_{i\in I}$ (resp. $\{\alpha_i^\vee\}_{i\in I}$), such that $\langle\alpha_i,\alpha_j^\vee\rangle=a_{ji}$, and constructs a Lie superalgebra $\tilde{\mathfrak{g}}(A,\tau)$ on generators $e_i, f_i, \mathfrak{h}$, subject to relations:

$$[\mathfrak{h},\mathfrak{h}]=0, \qquad [h,e_i]=\langle\alpha_i,h\rangle e_i, \qquad [h,f_i]=-\langle\alpha_i,h\rangle f_i, \quad \text{for } i\in I, h\in\mathfrak{h}, \quad [e_i,f_j]=\delta_{ij}\alpha_i^\vee,$$
$$p(e_i)=p(f_i)=\bar{1} \quad \text{if } i\in\tau, \qquad p(e_i)=p(f_i)=\bar{0} \quad \text{if } i\notin\tau, \qquad p(\mathfrak{h})=\bar{0}.$$

Then $\mathfrak{g}(A,\tau)=\tilde{\mathfrak{g}}(A,\tau)/J=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, where J is the maximal ideal of $\tilde{\mathfrak{g}}(A,\tau)$, intersecting $\mathfrak{h}$ trivially, and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the subalgebra generated by the images of the e_i's (resp. f_i's). We obtain the triangular decomposition $\mathfrak{g}(A)=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$.

Let Δ be the set of roots of $\mathfrak{g}(A)$, i.e. $\Delta=\{\alpha\in\mathfrak{h}^*\mid\alpha\neq0\ \&\ \mathfrak{g}_\alpha\neq0\}$, $\Delta_+=\{\alpha\in\mathfrak{h}^*\mid\mathfrak{n}_\alpha\neq0\}$, $\Delta_-=\{\alpha\in\mathfrak{h}^*\mid\mathfrak{n}_{-,\alpha}\neq0\}$. One has $\Delta=\Delta_+\sqcup\Delta_-$, $\Delta_-=-\Delta_+$.

We say that a simple root α_i is *even* (resp., *odd*) if $i\notin\tau$ (resp., $i\in\tau$) and that α_i is *isotropic* if $a_{ii}=0$. One readily sees that if $i\in\tau$ (i.e., e_i, f_i are odd), then $[e_i,e_i],[f_i,f_i]\in J$ iff $a_{ii}=0$. Therefore for a simple root α one has $2\alpha\in\Delta$ iff α is a non-isotropic and odd.

Note that, multiplying the i-th row of the matrix A by a non-zero number corresponds to multiplying e_i and $\alpha_i^\vee$ by this number, thus giving an isomorphic Lie superalgebra. Hence we may assume from now on that $a_{ii}=2$ or 0 for all $i\in I$.

1.1.1. We consider the case when the Cartan matrix $A=(a_{ij})$ is such that

(1) $a_{ii}\in\{0,2\}$ for all $i\in I$ and $a_{ij}=0$ forces $a_{ji}=0$;
(2) if $i\notin\tau$, then $a_{ii}=2$ and $a_{ij}\in\mathbb{Z}_{\leqslant0}$ for $j\neq i$;
(3) if $i\in\tau$ and $a_{ii}=2$, then $a_{ij}\in2\mathbb{Z}_{\leqslant0}$ for $j\neq i$.