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The isomorphism problem for universal enveloping algebras of nilpotent Lie algebras ☆

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ABSTRACT

In this paper we study the isomorphism problem for the universal enveloping algebras of nilpotent Lie algebras. We prove that if the characteristic of the underlying field is not 2 or 3, then the isomorphism type of a nilpotent Lie algebra of dimension at most 6 is determined by the isomorphism type of its universal enveloping algebra. Examples show that the restriction on the characteristic is necessary.

1. Introduction

In this paper we examine the isomorphism problem for universal enveloping algebras of Lie algebras. It is known that two non-isomorphic Lie algebras may have isomorphic universal enveloping algebras; see for instance [RU, Example A]. All such known examples require that the characteristic of the underlying field is a prime. In this paper we focus on nilpotent Lie algebras and prove the following main result.

Theorem 1.1. *The isomorphism type of a nilpotent Lie algebra of dimension at most* 6 *is determined by the isomorphism type of its universal enveloping algebra over any field of characteristic not* 2 *nor* 3.

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Theorem 1.1 is a consequence of Theorems 3.1 and 4.1. As shown by examples in Sections 3 and 4, the requirement about the characteristic of the underlying field is necessary. In addition to proving Theorem 1.1, we classify the possible isomorphisms between the universal enveloping algebras of nilpotent Lie algebras of dimension at most 5 over an arbitrary field, and those of dimension 6 over fields of characteristic different from 2 (see Theorems 3.1 and 4.1).

Little progress has ever been made on the isomorphism problem for universal enveloping algebras. For a Lie algebra L, let $U(L)$ denote its universal enveloping algebra (see Section 2 for the definitions). Several invariants of L are known to be determined by $U(L)$. For instance, if L is finite-dimensional, then the (linear) dimension of L coincides with the Gelfand–Kirillov dimension of $U(L)$ (see [KL]). More recently Riley and Usefi [RU] proved that the nilpotence of L is determined by $U(L)$ and, for a nilpotent L, the nilpotency class of L can be determined using $U(L)$. Moreover the isomorphism type of $U(L)$ determines the isomorphism type of the graded algebra $\mathrm{Gr}(L)$ associated with the lower central series of L (see Section 2 for the definitions). Malcolmson [M] showed that if L is a 3-dimensional simple Lie algebra over a field of characteristic not 2, then L is determined by $U(L)$ up to isomorphism. Later Chun, Kajiwara, and Lee [CKL] generalized Malcolmson's result to the class of all Lie algebras with dimension 3 over fields of characteristic not 2.

By proving Theorem 1.1, we verify that the isomorphism problem for universal enveloping algebras has a positive solution in the class of nilpotent Lie algebras with dimension at most 6 over fields of characteristic different from 2 and 3. The proof of this result relies on the classification of nilpotent Lie algebras with dimension at most 6. The classification of such Lie algebras of dimension at most 5 has been known for a long time over an arbitrary field. In dimension 6, several classifications have been published, but they were often incorrect, and they usually only treated fields of characteristic 0. Recently de Graaf [dG] published a classification of 6-dimensional nilpotent Lie algebras over an arbitrary field of characteristic not 2. As the classification by de Graaf has been obtained making heavy use of computer calculations, and was checked by computer for small fields [Sch], we consider this classification as the most reliable in the literature. The reason we do not treat 6-dimensional nilpotent Lie algebras over fields of characteristic 2 is that, in this case, we do not know of a similarly reliable classification.

Our strategy in proving Theorem 1.1 is to determine all pairs of nilpotent Lie algebras L_1, L_2 with dimension at most 6, such that the graded algebras $\mathrm{Gr}(L_1)$ and $\mathrm{Gr}(L_2)$ associated with the lower central series are isomorphic. We know from [RU] that this is a necessary condition for the isomorphism $U(L_1) \cong U(L_2)$. Such pairs can be read off from the list of nilpotent Lie algebras with dimension at most 6 in [dG]. Next, for all such pairs, we either argue that $U(L_1)$ cannot be isomorphic to $U(L_2)$, or we exhibit an explicit isomorphism between $U(L_1)$ and $U(L_2)$. Initially, computer experiments played a role in determining the isomorphisms between universal enveloping algebras of nilpotent Lie algebras. Recent work by Eick [E] describes a practical algorithm to decide isomorphism between finite-dimensional nilpotent associative algebras. Her algorithm was implemented in the ModIsom package [MI] of the GAP computational algebra system [GAP] and the implementations work for algebras of dimensions up to about 100 over small finite fields. We used this implementation to decide isomorphisms between the finite-dimensional, nilpotent quotients $\Omega(L)/\Omega^k(L)$ of the augmentation ideals $\Omega(L)$ of $U(L)$ (see Section 2 for notation). We remark here an interesting observation. If L is nilpotent of class c then based on our calculations the isomorphism type of the quotient $\Omega(L)/\Omega^{c+1}(L)$ determines the isomorphism type of L. So, the question remains whether $\Omega(L)/\Omega^{c+1}(L)$ determines the isomorphism type of L in all dimensions.

2. Preliminaries

In this section we summarize some important facts about universal enveloping algebras of Lie algebras; see [D] for a more detailed background. We assume from now on that Lie algebras are finite-dimensional, even though most of the results referred to in this section hold for a larger class of Lie algebras.