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The Zariski–Lipman conjecture for complete intersections

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ABSTRACT

The tangential branch locus $B^t_{X/Y} \subset B_{X/Y}$ is the subset of points in the branch locus where the sheaf of relative vector fields $T_{X/Y}$ fails to be locally free. It was conjectured by Zariski and Lipman that if V/k is a variety over a field k of characteristic 0 and $B^t_{V/k} = \emptyset$, then V/k is smooth ($=$ regular). We prove this conjecture when V/k is a locally complete intersection. We prove also that $B^t_{V/k} = \emptyset$ implies $\operatorname{codim}_X B_{V/k} \leqslant 1$ in positive characteristic, if V/k is the fibre of a flat morphism satisfying generic smoothness.

1. Introduction

Let $\pi : X \to Y$ be a morphism of noetherian schemes which is locally of finite type, $\Omega_{X/Y}$ its sheaf of Kähler differentials, and $T_{X/Y} = Hom_{\mathcal{O}_X}(\Omega_{X/Y}, \mathcal{O}_X)$ the sheaf of relative tangent vector fields. We have the inclusion of the tangential branch locus in the branch locus

$$B^t_\pi = \{x \in X \mid T_{X/Y,x} \text{ is not free}\} \subset B_\pi = B_{X/Y} = \{x \in X \mid \Omega_{X/Y,x} \text{ is not free}\}.$$

Define as in [5, Definitions 17.1.1 and 17.3.1] a morphism π to be formally smooth at a point x in X if the induced map of local rings $\mathcal{O}_{Y,\pi(x)} \to \mathcal{O}_{X,x}$ is formally smooth, and that π is smooth at x if it is locally finitely presented and formally smooth; say also that π is smooth if it is smooth at all points in X. In the light of the fact that the Jacobian criterion, namely that $B_\pi = \emptyset$, goes a long way to implying that the morphism π is smooth (Theorems 3.1 and 3.3), it is a natural to ask, with Zariski and Lipman [12], what are the implications of $B^t_\pi = \emptyset$? The example $X = \operatorname{Spec} A[x]/(x^2) \to Y = \operatorname{Spec} A$, i.e. the scheme of dual numbers over a commutative ring A, shows that if we want π

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to be smooth, the condition $B^t_\pi = \emptyset$ needs at least to be supplemented with the condition that the rank of $T_{X/Y}$ equals the relative dimension at each point in X, which can be imposed by assuming that X/Y is smooth at generic points in X. It is a remarkable fact that although $T_{X/Y}$ cannot even directly detect torsion in $\Omega_{X/Y}$, it turns out that these conditions combined imply $B_\pi = \emptyset$ (and hence imply that π is smooth) in interesting cases in characteristic 0. Already the result that $B^t_{V/k} = 0$ implies smoothness when V/k is a curve over a field of characteristic 0, due to Lipman [12], is, I think, quite surprising and non-trivial (see Proposition 4.4). In positive characteristic it is easy to see that smoothness at points of height $\leqslant 1$ does not follow from $B^t_\pi = \emptyset$, so one could perhaps add the assumption $\operatorname{codim}_X B_\pi \geqslant 2$; but this is still not enough. What is needed is a condition on the discriminant locus $D_\pi = \pi(B_\pi)$. Before the main results are presented we describe some terminology.

Generalities. All schemes are assumed to be noetherian and we use the notation in EGA, but see also [13, §5] and [8]. The height $\operatorname{ht}_X(x)$ of a point x in X is the same as the Krull dimension of the local ring $\mathcal{O}_{X,x}$ at x, and the dimension of X is defined as $\dim X = \sup\{\operatorname{ht}(x) \mid x \in X\}$. The dimension at a point x in X is

$$\dim_x X = \sup\bigl\{\operatorname{ht}(x_1) \bigm| x_1 \in X \text{ and } x \text{ specialises to } x_1\bigr\};$$

see [4, Proposition 5.1.4]. A point x in a subset T of X is *maximal* if for each point y in T that belongs to the closure $\{x\}^-$ of $\{x\}$ (in other words, x specialises to y (see [8, p. 93])), we have $\operatorname{ht}(x) \leqslant \operatorname{ht}(y)$. That is, if $x' \in T$ specialises to x, and $\operatorname{ht}(x') \leqslant \operatorname{ht}(x)$, then $x' = x$. Denote by $\operatorname{Max}(T)$ the set of maximal points of T, so $\operatorname{Max}(X)$ consists of points of height 0. A property on X is *generic* if it holds for all points in $\operatorname{Max}(X)$. Put

$$\operatorname{codim}^+_X T = \sup\bigl\{\operatorname{ht}(x) \bigm| x \in \operatorname{Max}(T)\bigr\},$$
$$\operatorname{codim}^-_X T = \inf\bigl\{\operatorname{ht}(x) \bigm| x \in \operatorname{Max}(T)\bigr\},$$

so $\operatorname{codim}^-_X T \leqslant \operatorname{ht}(x) \leqslant \operatorname{codim}^+_X T$ when $x \in \operatorname{Max}(T)$. If T is the empty set, put $\operatorname{codim}^+_X T = -1$ and $\operatorname{codim}^-_X T = \infty$, since we are interested in lower and higher bounds on $\operatorname{codim}^\pm_X T$, respectively. For a coherent $\mathcal{O}_X$-module M, the stalk at a point x is denoted M_x and we put $\operatorname{depth}_T M = \inf\{\operatorname{depth} M_x \mid x \in T\}$. The fibre X_y over a point y in Y is the fibre product $\operatorname{Spec} k_{Y,y} \times_Y Y$, where $k_{Y,y}$ is the residue field at y. We define the *relative dimension* $d_{X/Y,x}$ of π at a point $x \in X$ as the infimum of the dimension of the vector space of Kähler differentials at all maximal points ξ that specialise to x, i.e.

$$d_{X/Y,x} = \inf\bigl\{\dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{X/Y,\xi} \bigm| x \in \{\xi\}^-,\ \xi \in \operatorname{Max}(X)\bigr\}.$$

To understand this number it is useful recall that

$$\begin{aligned}\dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{X/Y,\xi} &= \dim_{k_{X_{\pi(\xi)},\xi}} \Omega_{X_{\pi(\xi)}/k_{Y,\pi(\xi)}} \\ &= \dim_{k_{X,\xi}} k_{X,\xi} \otimes_{\mathcal{O}_{X,\xi}} \Omega_{\mathcal{O}_{X,\xi}/\mathcal{O}_{Y,\pi(\xi)}};\end{aligned}$$

see Proposition 2.1 for the first equality, but note that in general the numbers $d_{X/Y,x}$ and $\dim_x X_{\pi(x)}$ are not equal. On the other hand, if π is flat at x, then $\dim_x X_{\pi(x)} = \dim \mathcal{O}_{X,x} - \dim \mathcal{O}_{Y,\pi(x)}$, and if moreover π is smooth at all points $\xi \in \operatorname{Max}(X)$ that specialise to x, then $d_{X/Y,x} = \dim_x X_{\pi(x)}$.

Recall also (this is an easy extension of [8, Chapter II, Lemma 8.9]):

(∗) a coherent $\mathcal{O}_X$-module M is free at a point x if M_ξ is free of rank equal to $\dim_{k_{X,x}} k_{X,x} \otimes_{\mathcal{O}_{X,x}} M_x$ for each $\xi \in \operatorname{Max}(X)$ that specialises to x.