Journal of Algebra 337 (2011) 369–384

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Singularities of duals of Grassmannians

Frédéric Holweck

Laboratoire M3M, Université de Technologie de Belfort-Montbéliard, 90010 Belfort Cedex, France

ARTICLE INFO

Article history:
Received 24 October 2010
Available online 7 May 2011
Communicated by Jean-Yves Thibon

Keywords:
Algebraic geometry
Projectively dual variety
Grassmannian
Secant varieties
Singular locus
Hyperdeterminant
Second fundamental form
Representation of semi-simple Lie algebras

ABSTRACT

Let $X \subset \mathbb{P}^N$ be a smooth irreducible nondegenerate projective variety and let $X^* \subset \mathbb{P}^{N}$ denote its dual variety. The locus of bitangent hyperplanes, i.e. hyperplanes tangent to at least two points of X, is a component of the singular locus of X^*. In this paper we provide a sufficient condition for this component to be of maximal dimension and show how it can be used to determine which dual varieties of Grassmannians are normal. That last part may be compared to what has been done for hyperdeterminants by J. Weyman and A. Zelevinsky (1996) in [23].

1. Introduction

Let $X = \mathbb{P}^n \times \mathbb{P}^n \subset \mathbb{P}^{(n+1)^2-1}$ be the Segre embedding of the product of two projective spaces of dimension n. The variety X corresponds to the projectivization of the variety of rank one matrices embedded in the projectivization of the space of $(n+1) \times (n+1)$ matrices. It is well known its dual variety (the variety of tangent hyperplanes, see below for the definition), denoted by X^*, can be identified with the variety of rank at most n matrices. Up to multiplication by a nonzero scalar, the equation defining $X^* \subset \mathbb{P}^{(n+1)\times(n+1)-1*}$ is the determinant. That point leads to a higher dimensional generalization of the determinant, called *hyperdeterminant*, which was first introduced by Cayley (1840) and rediscovered by Gelfand, Kapranov and Zelevinsky (1992). In [6,7] the authors define the hyperdeterminant of format $(k_1+1) \times \cdots \times (k_r+1)$ by the equation (up to scale) of the dual variety of $X = \mathbb{P}^{k_1} \times \cdots \times \mathbb{P}^{k_r} \subset \mathbb{P}^{(k_1+1)\times\cdots\times(k_r+1)-1}$. When the dual variety X^* is not a hypersurface, the corresponding hyperdeterminant is defined to be zero.

E-mail address: frederic.holweck@utbm.fr.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.04.023

Let $X \subset \mathbb{P}(V)$ be a projective variety and let $\tilde{T}_x X$ denote the embedded tangent space of X at a point $x \in \mathrm{Sm}(X)$ (smooth points of X). Define the *dual variety* X^* by

$$X^* = \overline{\{H \in \mathbb{P}(V^*) \mid \exists x \in \mathrm{Sm}(X) \text{ such that } \tilde{T}_x X \subset H\}} \subset \mathbb{P}(V^*)$$

The biduality theorem $(X^*)^* = X$ (true in characteristic zero) implies that the original variety can be reconstructed from its dual variety. Thus geometric invariants of X^* reflect in geometric properties of X. The dimension, degree and singularities of X^* carry meaningful information about the hyperplane sections of X (see [24]). These invariants have been studied for hyperdeterminants. In [6] a condition is given to decide whether or not the hyperdeterminant of a given format is nonzero (i.e. the dual of the Segre embedding is actually a hypersurface), moreover in the same paper the authors give a combinatorial formula to compute the degree of a given hyperdeterminant. They also conjectured that there is only one hyperdeterminant whose corresponding hypersurface is regular in codimension one, i.e. $\mathrm{codim}_{X^*} \mathrm{Sing}(X^*) \geqslant 2$, and this hyperdeterminant is of format $(2, 2, 2)$. In other words $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ is the only Segre product of at least three projective spaces whose dual variety is a normal hypersurface. That conjecture was proved by Weyman and Zelevinsky in [23].

Let $G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$ denote the Grassmannian of k-planes in $V = \mathbb{C}^n$, $k \leqslant n-k$, embedded through the Plücker map. Its dual variety is a hypersurface except if $k = 2$ and n is odd [14]. The degree of $G(k,n)^*$ has been studied in [17]. However the study of $\mathrm{Sing}(G(k,n)^*)$ has not been carried out so far. In this article we answer the question of the normality of the duals of Grassmannian varieties. The case of the Grassmannian of 2-planes is known and similar to the Segre product of two projective spaces. The variety $G(2,n) \subset \mathbb{P}(\Lambda^2\mathbb{C}^n)$ corresponds to the projectivization of the rank 2 skew-symmetric matrices and its dual is identified with degenerate skew-symmetric matrices. Like for the determinant, the singular locus of the degenerate skew-symmetric matrices is regular in codimension 1 and arithmetically Cohen Macaulay (see [12]). This proves that $G(2,n)^*$ is normal.

For $k \geqslant 3$ the dual variety $G(k,n)^*$ is a hypersurface. Thus $G(k,n)^*$ will be normal if and only if $G(k,n)^*$ is regular in codimension one. This will be the main result of this article:

Theorem 1. *Let $X = G(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$, with $k \geqslant 3$. The dual variety $G(k,n)^*$ is normal if and only if X is one of the following*:

$$G(3,6) \subset \mathbb{P}^{19}, \quad G(3,7) \subset \mathbb{P}^{34}, \quad G(3,8) \subset \mathbb{P}^{55}$$

Remark 1.1. Like for hyperdeterminants the general pattern is the following: the variety X^* has a singular locus of codimension one and the only exceptions come from group actions with finite numbers of orbits.

The proof is based on the calculation of the dimension of $\sigma_2(X)^*$, the dual of the secant variety of X, which is always a component of $\mathrm{Sing}(X^*)$. It turns out that this component appears in the decomposition of the singular locus of hyperdeterminants by [23]. In their paper it corresponds to the general double point locus or node locus denoted by $\nabla_{node}(\emptyset)$ (i.e. the set of hyperplane having more than one point of tangency on X). An other component of interest is the cusp locus (i.e. set of hyperplanes defining degenerate quadrics). The geometrical meaning of $\nabla_{node}(\emptyset)$ is not emphasized in [23] when they calculate the dimension of this component. Here in the contrary we mainly use geometric arguments to calculate the dimension of $\sigma_2(X)^*$ in the general case. Let $\hat{T}^{(2)}_x X$ be the (cone over the) second osculating space, i.e. the linear span of second osculating spaces of smooth curves $x(t) \subset X$ with $x(0) = x$. In Section 3 we prove:

Proposition 1. *Let $X \subset \mathbb{P}(V)$ be a smooth projective variety of dimension n. Assume X^* is a hypersurface. Suppose for a general pair of point $(x, y) \in X \times X$ we have $\hat{T}^{(2)}_x X \cap \hat{T}_y X = \{0\}$, then* $\mathrm{codim}_{X^*} \sigma_2(X)^* = 1$. *In particular X^* is not normal.*