Journal of Algebra 334 (2011) 109–149

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A generalization of Gabriel's Galois covering functors and derived equivalences ☆

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ARTICLE INFO

Article history:
Received 19 April 2010
Communicated by Michel Broué

MSC:
18A32
16B50
16G20

Keywords:
Covering functors
Orbit categories
Derived equivalences

ABSTRACT

Let G be a group acting on a category $\mathcal{C}$. We give a definition for a functor $F : \mathcal{C} \to \mathcal{C}'$ to be a G-covering and three constructions of the orbit category $\mathcal{C}/G$, which generalizes the notion of a Galois covering of locally finite-dimensional categories with group G whose action on $\mathcal{C}$ is free and locally bonded defined by Gabriel. Here $\mathcal{C}/G$ is defined for any category $\mathcal{C}$ and we do not require that the action of G is free or locally bounded. We show that a G-covering is a universal "G-invariant" functor and is essentially given by the canonical functor $\mathcal{C} \to \mathcal{C}/G$. By using this we improve a covering technique for derived equivalences. Also we prove theorems describing the relationships between smash product construction and the orbit category construction by Cibils and Marcos (2006) without the assumption that the G-action is free. The orbit category construction by a cyclic group generated by an auto-equivalence modulo natural isomorphisms (e.g., the construction of cluster categories) is justified by a notion of the "colimit orbit category". In addition, we give a presentation of the orbit category of a category with a monoid action by a quiver with relations, which enables us to calculate many examples.

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Introduction

Throughout this paper G is a group (except for Sections 8, 9) and $\Bbbk$ is a commutative ring, and all categories, functors and algebras are assumed to be $\Bbbk$-linear unless otherwise stated. (Here a category is called a $\Bbbk$-linear category (or a $\Bbbk$-category for short) if its morphism sets are $\Bbbk$-modules and its compositions are $\Bbbk$-bilinear, and we do not require that it is additive.) A pair $(\mathcal{C}, A)$ of a category

☆ This work is partially supported by Grant-in-Aid for Scientific Research (C) 17540036 from JSPS.
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doi:10.1016/j.jalgebra.2011.03.002

$\mathcal{C}$ and a group homomorphism $A : G \to \mathrm{Aut}(\mathcal{C})$ is called a category with a *G-action* or a *G-category*, where $\mathrm{Aut}(\mathcal{C})$ is the group of automorphisms of $\mathcal{C}$ (not the group of auto-equivalences of $\mathcal{C}$ modulo natural isomorphisms). We set $A_\alpha := A(\alpha)$ for all $\alpha \in G$. If there is no confusion we always (except for Sections 8, 9) denote G-actions by the same letter A, and simply write $\mathcal{C} = (\mathcal{C}, A)$, and further we usually write $\alpha x := A_\alpha x$, $\alpha f := A_\alpha f$ for all $x \in \mathcal{C}$ and all morphisms f in $\mathcal{C}$.

Classical covering technique. Let $F : \mathcal{C} \to \mathcal{C}'$ be a functor with $\mathcal{C}$ a G-category. The classical setting of covering technique (see e.g. [13]) required the following conditions:

(1) $\mathcal{C}$ is *basic* (i.e., $x \neq y \Rightarrow x \not\cong y$);
(2) $\mathcal{C}$ is *semiperfect* (i.e., $\mathcal{C}(x, x)$ is a local algebra, $\forall x \in \mathcal{C}$);
(3) G-action is *free* (i.e., $1 \neq \forall\alpha \in G$, $\forall x \in \mathcal{C}$, $\alpha x \neq x$); and
(4) G-action is *locally bounded* (i.e., $\forall x, y \in \mathcal{C}$, $\{\alpha \in G \mid \mathcal{C}(\alpha x, y) \neq 0\}$ is finite).

But these assumptions made it very inconvenient to apply the covering technique to usual additive categories such as the bounded homotopy category $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, R)$ of finitely generated projective modules over a ring R or even the module category $\mathrm{Mod}\, R$ of R because these categories do not satisfy the condition (2) and hence we have to construct the full subcategory of indecomposable objects, which destroys additional structures like a structure of a triangulated category; and to satisfy the condition (1) we have to choose a complete set of representatives of isoclasses of objects that should be stable under the G-action, which is not so easy in practice; and also the condition (3) is difficult to check in many cases, e.g., even in the case when we use G-actions on the category $\mathcal{K}^{\mathrm{b}}(\mathrm{prj}\, R)$ or on $\mathrm{Mod}\, R$ induced from that on R. These made the proof of the main theorem of a covering technique for derived equivalences in [1] unnecessarily complicated and prevented wider applications. The first purpose of this paper is to generalize the covering technique to remove all these assumptions.

Orbit categories and covering functors. Recall that to define a so-called "root category" $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, H)/[2]$ of a hereditary algebra H over a field in Happel [16] or in Peng and Xiao [21] we needed a generalization that removes at least conditions (1) and (2). It seems, however, even such a simple generalization was not found explicitly in the literature for a long time. The definition of root categories given in [21] works only for itself, and does not give a general definition of orbit categories. Nevertheless, their definition was useful to show that the obtained orbit category is a triangulated category. This gave us one of the motivations to start this work. Recently general definitions of orbit categories were given in [9] by Cibils and Marcos (let us denote it by $\mathcal{C}/_1 G$) and in [19] by Keller (in the case that G is cyclic, let us denote it by $\mathcal{C}/_2 G$). But we still did not understand the relationship between the notion of covering functors by Gabriel [13] and the orbit categories defined by them. We wanted to generalize Gabriel's covering technique as much as possible. To this end it was necessary to generalize the definition of a covering functor. In the classical setting the first condition for a functor F to be a (Galois) covering functor (with group G) is that $F = FA_\alpha$ for all $\alpha \in G$. This leads us naturally to a definition of an *invariance adjuster*, a family of natural isomorphisms $\phi := (\phi_\alpha : F \to FA_\alpha)_{\alpha \in G}$ (see Definition 1.1). The pair (F, ϕ) is called a (*right*) *G-invariant* functor, further which is called a *G-covering* functor if F is a dense functor such that both

$$F^{(1)}_{x,y} : \bigoplus_{\alpha \in G} \mathcal{C}(\alpha x, y) \to \mathcal{C}'(Fx, Fy), \quad (f_\alpha)_{\alpha \in G} \mapsto \sum_{\alpha \in G} F(f_\alpha) \cdot \phi_{\alpha,x}, \quad \text{and}$$

$$F^{(2)}_{x,y} : \bigoplus_{\beta \in G} \mathcal{C}(x, \beta y) \to \mathcal{C}'(Fx, Fy), \quad (f_\beta)_{\beta \in G} \mapsto \sum_{\beta \in G} \phi^{-1}_{\beta,y} \cdot F(f_\beta)$$

are isomorphisms of $\Bbbk$-modules for all $x, y \in \mathcal{C}$. In fact, it is enough to require that either $F^{(1)}_{x,y}$ or $F^{(2)}_{x,y}$ is an isomorphism for each $x, y \in \mathcal{C}$. Roughly speaking the definition of $\mathcal{C}' := \mathcal{C}/_1 G$ (resp. $\mathcal{C}' := \mathcal{C}/_2 G$) yields by setting all the $F^{(1)}_{x,y}$ (resp. $F^{(2)}_{x,y}$) to be the identities. In this paper we give a "left–right symmetric" construction of the orbit category $\mathcal{C}/G$ of $\mathcal{C}$ by G, which is a direct modification of Gabriel's

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