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Symmetric groups are determined by their character degrees ☆

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ABSTRACT

Let G be a finite group. Let $X_1(G)$ be the first column of the ordinary character table of G. In this paper, we will show that if $X_1(G) = X_1(S_n)$, then $G \cong S_n$. As a consequence, we show that S_n is uniquely determined by the structure of the complex group algebra $\mathbb{C}S_n$.

1. Introduction and notations

All groups considered are finite and all characters are complex characters. Let G be a group and let $Irr(G) = \{\chi_1, \chi_2, \dots, \chi_k\}$ be the set of all irreducible characters of G. Put $n_i = \chi_i(1)$. We say that $(n_1, n_2, \dots, n_k)$ is the *degree pattern* of G. Let $cd(G) = \{\chi(1) \mid \chi \in Irr(G)\}$ be the set of all irreducible character degrees of G. Following [2], let $X_1(G)$ be the first column of the ordinary character table of G. By a suitable re-ordering of the rows in the character table of G, we can see that $X_1(G)$ coincides with the degree pattern $(n_1, n_2, \dots, n_k)$ of G. We also consider $X_1(G)$ as a multiset consisting of character degrees of G counting multiplicities. Since $|G| = \sum_{\chi \in Irr(G)} \chi(1)^2$, the order of G is known given $X_1(G)$. There are examples showing that non-isomorphic groups may have the same character table and so the first column of their character tables coincide. Using the classification of finite simple groups, it is easy to see that non-abelian simple groups are uniquely determined by their character tables. It was shown by Nagao [14] that the symmetric groups S_n are also uniquely determined by their character tables. In [16], we know that the alternating group A_n of degree at least 5, and the sporadic

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simple groups are uniquely determined by the first column of their character tables. In this paper, we will prove a similar result for the symmetric groups.

Theorem 1.1. *Let G be a finite group. If* $X_1(G) = X_1(S_n)$*, then* $G \cong S_n$*.*

This gives a positive answer to [2, Question 126]. Let $\mathbb{C}$ be the complex number field and let G be a group. Denote by $\mathbb{C}G$ the group algebra of G over $\mathbb{C}$. Let G_i, $i = 1, 2$, be groups. By Molien's Theorem [2, Theorem 2.13] we know that $\mathbb{C}G_1 \cong \mathbb{C}G_2$ if and only if $X_1(G_1) = X_1(G_2)$. Therefore, knowing the first column of the character table of a group G is equivalent to knowing the structure of the group algebra $\mathbb{C}G$. It is known that $\mathbb{C}G$ allows us to recognize the Frobenius groups or the p-nilpotent groups [2, Corollaries 10.11 and 10.27]. Now Theorem 1.1 yields.

Corollary 1.2. *Let G be a group. If* $\mathbb{C}G \cong \mathbb{C}S_n$*, then* $G \cong S_n$*.*

We should mention that Brauer's Problem 1 (see [3]) which asks the following: What are the possible degree patterns of finite groups? Little is known about this problem. Now Corollary 1.2 says that there is exactly one isomorphism type of the group algebra with a degree pattern as that of the symmetric groups.

We now outline our argument for the proof of Theorem 1.1. Assume that $X_1(G) = X_1(S_n)$. We first observe that $|G : G'| = 2$, $|G| = n!$, $k(G) = k(S_n)$ and $cd(G) = cd(S_n)$, where $k(G)$ denotes the number of conjugacy classes of G. The result is trivial when $n \leqslant 4$. Hence we will assume that $n \geqslant 5$. Next we will show that G' is perfect, that is $G' = G''$, by applying [8, Lemma 12.3]. Choose $M \leqslant G'$ be a normal subgroup of G so that G'/M is a chief factor of G. As $|G : G'| = 2$ and G'/M is non-abelian, we deduce that $G'/M \cong S^k$, where S is a non-abelian simple group and k is at most 2. We proceed to show that G'/M must be a simple group that is $k = 1$. This is done by applying Theorem 3.3. We now deduce that either G/M is an almost simple group with socle G'/M or $G/M \cong G'/M \times \mathbb{Z}_2$. We now apply Theorem 3.1 which asserts that if H is an almost simple group and $cd(H) \subseteq cd(S_n)$, $n \geqslant 5$, then the socle of H must be isomorphic to A_n, to show that $G'/M \cong A_n$. Assume that $n \neq 6$. By comparing the orders, $G \cong S_n$ or $G \cong A_n \times \mathbb{Z}_2$. Finally, using the fact that G and S_n have the same number of irreducible characters, we can eliminate the latter case. Thus G must be isomorphic to S_n. In the exceptional case, we have $|Out(A_6)| = 4$. In this case, G is one of the following groups: $A_6 \times \mathbb{Z}_2$, $PGL_2(9) \cong A_6.2_2$, $M_{10} \cong A_6.2_3$ or S_6. Using [5], we conclude that $G \cong S_6$. We remark that this argument is based on Huppert's method given in [7]. This method is used to verify the Huppert Conjecture which states that non-abelian simple groups are determined by their sets of character degrees (see [7,17]).

Here are some notations. If $cd(G) = \{s_0, s_1, \ldots, s_t\}$, where $1 = s_0 < s_1 < \cdots < s_t$, then we define $d_i(G) = s_i$ for all $1 \leqslant i \leqslant t$. Then $d_i(G)$ is the ith smallest degree of the non-trivial character degrees of G. If n is an integer then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a group, we will write $\pi(G)$ instead of $\pi(|G|)$ to denote the set of all prime divisors of the order of G. Let $p(G) = max(\pi(G))$ be the largest prime divisor of the order of G and let $\rho(G) = \bigcup_{\chi \in Irr(G)} \pi(\chi(1))$ be the set of all primes which divide some irreducible character degrees of G. Finally, if $N \trianglelefteqslant G$ and $\theta \in Irr(N)$, then the inertia group of θ in G is denoted by $I_G(\theta)$. Other notations are standard.

2. Preliminaries

Let n be a positive integer. We call $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ a partition of n, written $\lambda \vdash n$, provided λ_i, $i = 1, 2, \ldots, r$ are integers, with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ and $\sum_{i=1}^{r} \lambda_i = n$. We collect the same parts together and write $\lambda = (\ell_1^{a_1}, \ell_2^{a_2}, \ldots, \ell_k^{a_k})$, with $\ell_i > \ell_{i+1} > 0$ for $i = 1, \ldots, k-1$; $a_i \neq 0$; and $\sum_{i=1}^{k} a_i \ell_i = n$. It is well known that the irreducible complex characters of the symmetric group S_n are parametrized by partitions of n. Denote by χ^λ the irreducible character of S_n corresponding to partition λ. The irreducible characters of the alternating group A_n are then obtained by restricting χ^λ to A_n. In fact, χ^λ is still irreducible upon restriction to the alternating group A_n if and only if λ is