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# Minimal fusion systems with a unique maximal parabolic

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ABSTRACT

We define minimal fusion systems in a way that every non-solvable fusion system has a section which is minimal. Minimal fusion systems can also be seen as analogs of Thompson's N-groups. In this paper, we consider a minimal fusion system $\mathcal{F}$ on a finite $p$-group $S$ that has a unique maximal $p$-local subsystem containing $N_{\mathcal{F}}(S)$. For an arbitrary prime $p$, we determine the structure of a certain (explicitly described) $p$-local subsystem of $\mathcal{F}$. If $p=2$, this leads to a complete classification of the fusion system $\mathcal{F}$.



## 1. Introduction

A pattern for the classification of finite simple groups was set by Thompson in [Th], where he gave a classification of all finite simple *N-groups*. These are non-abelian finite simple groups with the property that every $p$-local subgroup is solvable, for every prime $p$. Recall that a *$p$-local subgroup* of a finite group $G$ is the normalizer of a non-trivial $p$-subgroup of $G$. Thompson's work was generalized by Gorenstein and Lyons, Janko and Smith to *(N2)-groups*, that is to non-abelian finite simple groups all of whose 2-local subgroups are solvable. Recall here that, by the Feit–Thompson Theorem, every non-solvable group has even order.

N-groups play an important role, as every minimal non-solvable finite group is an N-group. Furthermore, every non-solvable group has a section which is an N-group. The respective properties hold also for (N2)-groups.

A new proof for the classification of (N2)-groups was given by Stellmacher in [St2]. It uses the amalgam method, which is a completely local method. Currently, Aschbacher is working on another new proof for the classification of (N2)-groups. His approach uses *saturated fusion systems* that were first introduced by Puig under the name of *full Frobenius categories*. Aschbacher's plan is to classify all *N-systems*, i.e. all saturated fusion systems $\mathcal{F}$ of characteristic 2-type such that the group $Mor_{\mathcal{F}}(P, P)$ is solvable, for every subgroup $P$ of $\mathcal{F}$. Here the use of the group theoretical concept of solvability fits

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with the definition of solvable fusion systems as introduced by Puig. However, this concept seems not general enough to ensure that N-systems play the same role in saturated fusion systems as N-groups in groups. Therefore, in our notion of *minimal fusion systems* introduced below, we find it necessary to use a concept of solvable fusion systems as defined by Aschbacher [A1, 15.1].

*For the remainder of the introduction let $p$ be a prime and $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $S$.* We adapt the standard terminology regarding fusion systems as introduced by Broto, Levi and Oliver [BLO]. For further basic definitions and notation we refer the reader to Section 2. Generic examples of saturated fusion systems are the fusion systems $\mathcal{F}_S(G)$, where $G$ is a finite group containing $S$ as a Sylow $p$-subgroup, the objects of $\mathcal{F}_S(G)$ are all subgroups of $S$, and the morphisms in $\mathcal{F}_S(G)$ between two objects are the injective group homomorphisms obtained by conjugation with elements of $G$.

**Definition 1.1.** The fusion system $\mathcal{F}$ is called minimal if $O_p(\mathcal{F}) = 1$ and $N_{\mathcal{F}}(U)$ is solvable for every fully normalized subgroup $U \neq 1$ of $\mathcal{F}$.

Here the fusion system $\mathcal{F}$ is *solvable*, if and only if $O_p(\mathcal{F}/R) \neq 1$, for every strongly closed subgroup $R \neq S$ of $\mathcal{F}$. This implies that indeed every minimal non-solvable fusion system is minimal in the sense defined above. Furthermore, every non-solvable fusion system has a section which is minimal. Therefore, minimal fusion systems play a similar role in saturated fusion systems as N-groups in groups. However, a classification of minimal fusion systems seems a difficult generalization of the original N-group problem. One reason is that in fusion systems the prime 2 does not play such a distinguished role as in groups. Therefore, we would like to treat minimal fusion systems also for odd primes as far as possible. Secondly, the notion of solvability in fusion systems is more general than the group theoretical notion. More precisely, although it turns out that every solvable fusion system is constrained and therefore the fusion system of a finite group, such a group can have certain composition factors that are non-abelian finite simple groups. Aschbacher showed in [A1] that these are all finite simple groups in which fusion is controlled in the normalizer of a Sylow $p$-subgroup. Furthermore, Aschbacher gives a list of these groups. Generic examples are the finite simple groups of Lie type in characteristic $p$ of Lie rank 1. For odd primes, Aschbacher's proof of these facts requires the complete classification of finite simple groups. For $p = 2$ they follow already from Goldschmidt's theorem on groups with a strongly closed abelian subgroup (see [Gold]).

In this paper, we use a concept which is an analog to the (abstract) concept of parabolics in finite group theory, where a *parabolic subgroup* is defined to be a $p$-local subgroup containing a Sylow $p$-subgroup. This generalizes the definition of parabolics in finite groups of Lie type in characteristic $p$. Suppose $S$ is a Sylow $p$-subgroup of a finite group $G$. It is a common strategy in the classification of finite simple groups and related problems to treat separately the case of a unique maximal (with respect to inclusion) parabolic containing $S$. In this case, one classifies as a first step a $p$-local subgroup of $G$ which has the pushing up property as defined in Section 6. In the remaining case, two distinct maximal parabolics containing $S$ form an amalgam of two groups that do not have a common normal $p$-subgroup. This usually allows an elegant treatment using the coset graph, and leads in the generic cases to a group of Lie type and Lie rank at least 2. The main result of this paper handles the fusion system configuration which loosely corresponds to the pushing up case in the N-group investigation. We next introduce the concept of a parabolic in fusion systems.

**Definition 1.2.**

- A subsystem of $\mathcal{F}$ of the form $N_{\mathcal{F}}(R)$ for some non-trivial normal subgroup $R$ of $S$ is called a *parabolic subsystem* of $\mathcal{F}$, or in short, a parabolic.
- A *full parabolic* is a parabolic containing $N_{\mathcal{F}}(S)$. It is called a *full maximal parabolic*, if it is not properly contained in any other parabolic subsystem of $\mathcal{F}$.

Thus, in this paper, we treat the case of a minimal fusion system having a unique full maximal parabolic. Note that this assumption is slightly more general than just supposing that a minimal fusion