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Observable subgroups of algebraic monoids

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ABSTRACT

A closed subgroup H of the affine, algebraic group G is called *observable* if G/H is a quasi-affine algebraic variety. In this paper we define the notion of an observable subgroup of the affine, algebraic *monoid* M. We prove that a subgroup H of G (the unit group of M) is observable in M if and only if H is closed in M and there are "enough" H-semiinvariant functions in $\Bbbk[M]$. We show also that a closed, normal subgroup H of G (the unit group of M) is observable in M if and only if it is closed in M. In such a case there exists a *determinant* $\chi : M \to \Bbbk$ such that $H \subseteq \ker(\chi)$. As an application, we show that in this case the *affinized quotient* $M/_{\mathrm{aff}}H$ of M by H is an affine algebraic monoid with unit group G/H.

1. Introduction

A closed subgroup H of the affine algebraic group G is called an *observable subgroup* if the homogeneous space G/H is a quasi-affine variety. Such subgroups have been researched extensively, notably by F. Grosshans, see [5] for a survey on this topic, and Theorem 2.12 below for other useful characterizations of observable subgroups. In [10] the authors presented the notion of an *observable action* of G on the affine variety X, together to its basic properties. In this paper we develop further the notion of observable actions. In particular, we investigate the situation where M is an affine *algebraic monoid* with unit group G, and H is a closed subgroup of G, such that the action of H on M by left multiplication is observable. In this case, we say that H is *observable in M*.

We describe now the organization of this paper.

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In Section 2 we provide the basic definitions and properties of observable actions and affinized quotients. In Section 3 we give several characterizations of observable subgroups. In Theorem 3.3 we deduce a number of useful consequences from the assumption that H is observable in M. In particular it follows that H is an observable subgroup of G, and that it is closed in M. In Theorem 3.4 we characterize the observable subgroups of M in terms of semiinvariants. In Theorem 3.5 we show that H is an observable subgroup of M if and only if H is the isotropy group of some vector $v \in V$ in some rational representation $\rho : M \to \mathrm{End}(V)$ of M. In the final section (Section 4) we use the results of the previous section to study the affinized quotient of an affine algebraic monoid by a closed normal subgroup. In Theorem 4.4 we show that if H is a closed, normal subgroup of G, closed in M, then H is observable in M. Whether this is true for nonnormal closed subgroups of G is an open question. See Remark 4.6. As an application, we show that the *affinized quotient* of an affine algebraic monoid M by a normal subgroup H, closed in M, is an algebraic monoid, with unit group G/H.

2. Preliminaries

Let $\Bbbk$ be an algebraically closed field. We work with affine algebraic varieties X over $\Bbbk$. An algebraic group is assumed to be a smooth, affine, group scheme of finite type over $\Bbbk$. If X is an affine variety over $\Bbbk$ we denote by $\Bbbk[X]$ the ring of regular functions on X. If $I \subset \Bbbk[X]$ is an ideal, we denote by $\mathcal{V}(I) = \{x \in X\colon\ f(x) = 0,\ \forall f \in I\}$. If $Y \subset X$ is a subset, we denote by $\mathcal{I}(Y) = \{f \in \Bbbk[X]\colon\ f(y) = 0, \forall y \in Y\}$. Morphisms $\varphi : X \to Y$ between affine varieties correspond to morphisms of algebras $\Bbbk[Y] \to \Bbbk[X]$, by $\varphi \mapsto \varphi^*$, $\varphi^*(f) = f \circ \varphi$. If X is irreducible we denote by $\Bbbk(X)$ the field of rational functions on X. If A is any integral domain we denote by $[A]$ its quotient field. Thus if X is an irreducible affine variety, then $\Bbbk(X) = [\Bbbk[X]]$.

Let G be an affine algebraic group and let X be an algebraic variety. A (*regular*) *action* of G on X is a morphism $\varphi : G \times X \to X$, denoted by $\varphi(g, x) = g \cdot x$, such that $(ab) \cdot x = a \cdot (b \cdot x)$ and $1 \cdot x = x$ for all $a, b \in G$ and $x \in X$. Since all the actions we work with are regular, we drop the adjective regular. The *orbit* of $x \in X$ is denoted by $\mathcal{O}(x) = \{g \cdot x\colon\ g \in G\}$.

If $G \times X \to X$ is a regular left action of G on X we consider the induced right action of G on $\Bbbk[X]$, defined as follows. If $f \in \Bbbk[X]$ and $g \in G$, then $(f \cdot g)(x) = f(gx)$. It is well known that G-stable closed subsets of X correspond to G-stable radical ideals of $\Bbbk[X]$. We say that $f \in \Bbbk[X]$ is *G-invariant* if $f \cdot g = f$ for any $g \in G$. The set of G-invariants ${}^{G}\Bbbk[X]$ forms a $\Bbbk$-subalgebra of $\Bbbk[X]$, possibly non-finitely generated. Analogous considerations can be made if we start with a right action $X \times G \to X$.

A *finite dimensional* (*rational*) *G-module* is a finite dimension $\Bbbk$-vector space V together with a left action of G on V by linear automorphisms. A right action of G on V defines, in a similar way, the notion of a *right G-module*.

Recall that an *algebraic monoid* M is an algebraic variety together with an associative product $m : M \times M \to M$ with neutral element 1, such that m is a morphism of algebraic varieties. We denote the *set of idempotent elements* of M by $E(M) = \{e \in M\colon\ e^2 = e\}$. We denote the *unit group* of M by $G(M) = \{g \in M\colon\ \exists g^{-1} \in M,\ g^{-1}g = gg^{-1} = 1\}$. It is known that $G(M)$ is an algebraic group, open in M (see [9] and [11]).

2.1. Characters of affine algebraic monoids

In this section we establish the basic facts about *extendible characters* for the case of linear algebraic monoids.

Definition 2.1. Let M be an algebraic monoid. A *character* of M is a morphism of algebraic monoids $M \to \Bbbk$. We denote the monoid of characters of M by

$$\mathcal{X}(M) = \big\{\chi \in \Bbbk[M]\colon\ \chi(ab) = \chi(a)\chi(b),\ \forall a, b \in M,\ \chi(1) = 1\big\}.$$

If $G = G(M)$, then restriction induces an injective morphism of (abstract) monoids $\mathcal{X}(M) \hookrightarrow \mathcal{X}(G)$.