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Asymmetry of Ext-groups

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ABSTRACT

In this paper, we will use techniques of noncommutative projective geometry to construct examples of algebras R over a field k not satisfying the following two types of symmetric behaviors of Ext-groups: (**EE**) For any pair of finitely generated R-modules (M,N), $\dim_k \operatorname{Ext}^i_R(M,N) < \infty$ for all $i \in \mathbb{N}$ if and only if $\dim_k \operatorname{Ext}^i_R(N,M) < \infty$ for all $i \in \mathbb{N}$. (**ee**) For any pair of finitely generated R-modules (M,N), $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N,M) = 0$ for all $i \gg 0$. In particular, and contrary to the commutative case, we give a simple example of a noncommutative noetherian Gorenstein (Frobenius) local algebra satisfying (**uac**) (uniform Auslander condition) but not satisfying (**ee**).

1. Introduction

1.1. Motivation

Throughout, we fix a base field k. Let R be an algebra over k. We denote by $\operatorname{mod} R$ the category of finitely generated right R-modules. For $M, N \in \operatorname{mod} R$, we defined in [16] the intersection multiplicity of M and N by $M \cdot N := (-1)^{\operatorname{GKdim} R - \operatorname{GKdim} M} \xi(M,N)$ where $\operatorname{GKdim} M$ is the Gelfand–Kirillov dimension of M and

$$\xi(M,N) := \sum_{i \in \mathbb{N}} (-1)^i \dim_k \operatorname{Ext}^i_R(M,N)$$

is the Euler form of M and N. In order for this definition to yield a good intersection theory, we must have the property $M \cdot N = N \cdot M$ for reasonably nice R, M, N. Although, for each i, $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Ext}^i_R(N,M)$ are very different in general (even if R is commutative), the property $M \cdot N = N \cdot M$

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holds in many situations. For example, over a commutative noetherian Gorenstein local algebra, this property is equivalent to the Serre's vanishing conjecture [11]. Moreover, for a noetherian connected graded algebra R, and finitely generated graded right R-modules M and N, it is often the case that $M \cdot N = N \cdot M$ as long as both $M \cdot N$ and $N \cdot M$ are well defined [9]. So the natural question is to ask whether $M \cdot N$ is well defined if and only if $N \cdot M$ is well defined (in the graded case). Since $M \cdot N$ is well defined if and only if (i) $\dim_k \operatorname{Ext}^i_R(M, N) < \infty$ for all $i \in \mathbb{N}$, and (ii) $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$, we ask whether R satisfies the following two types of symmetric behaviors of Ext-groups:

- (**EE**): for all $M, N \in \operatorname{mod} R$, $\dim_k \operatorname{Ext}^i_R(M, N) < \infty$ for all $i \in \mathbb{N}$ if and only if $\dim_k \operatorname{Ext}^i_R(N, M) < \infty$ for all $i \in \mathbb{N}$.
- (**ee**): for all $M, N \in \operatorname{mod} R$, $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$ if and only if $\operatorname{Ext}^i_R(N, M) = 0$ for all $i \gg 0$.

It is known that every commutative noetherian local algebra satisfies (**EE**) [11, Corollary 3.2]. On the other hand, it is easy to see that if R is a local algebra satisfying (**ee**), then R must be Gorenstein, so we focus on studying Gorenstein algebras in this paper. Presumably, Avramov and Buchweitz were the first people who studied the condition (**ee**), and they proved that every commutative complete intersection ring satisfies (**ee**) in [2]. The first example of a noetherian Gorenstein algebra not satisfying (**ee**) was given by Jorgensen and Sega in [6], which is even a commutative Frobenius local algebra. Later, a simpler but noncommutative (Frobenius) algebra not satisfying (**ee**) was given in [4]. (We thank Petter Andreas Bergh for pointing this out.) In this paper, we will use techniques of noncommutative projective geometry to construct noncommutative algebras not satisfying (**EE**) and those not satisfying (**ee**).

Related to the condition (**ee**), there is another condition (**uac**) (uniform Auslander condition) on a ring R:

- (**uac**): there is an integer $d_R \in \mathbb{N}$ such that, for all $M, N \in \operatorname{mod} R$, if $\operatorname{Ext}^i_R(M, N) = 0$ for all $i \gg 0$, then $\operatorname{Ext}^i_R(M, N) = 0$ for all $i > d_R$.

It was shown [5, Theorem 4.1] that every commutative noetherian Gorenstein local ring satisfying (**uac**) satisfies (**ee**). Contrary to the commutative case, we give a simple example of a noncommutative noetherian Gorenstein (Frobenius) local algebra satisfying (**uac**) but not satisfying (**ee**). (In fact, we will see that this happens quite frequently.)

Along the way toward the main results, we also prove that, for every FBN (fully bounded noetherian) AS-Gorenstein Koszul algebra, there is a bijection between isomorphism classes of point modules over A and those over A^o, extending [15, Theorem 6.3].

1.2. Noncommutative projective geometry

In this subsection, we review some of the language of noncommutative projective geometry which will be needed in this paper. We refer to [1] for details. Let A be a graded algebra. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules, and by $\operatorname{grmod} A$ the full subcategory of finitely generated graded right A-modules. Morphisms in $\operatorname{GrMod} A$ are A-module homomorphisms preserving degrees. The category of (finitely generated) graded left A-modules can be identified with $\operatorname{GrMod} A^o$ ($\operatorname{grmod} A^o$) where A^o is the opposite graded algebra of A.

For a vector space V over k, we denote by V^* the dual vector space of V. For a graded vector space $V \in \operatorname{GrMod} k$, we define the graded vector space dual $V^* \in \operatorname{GrMod} k$ by $(V^*)_i := (V_{-i})^*$ for $i \in \mathbb{Z}$. Moreover, for an integer $n \in \mathbb{Z}$, we define the truncation $V_{\geqslant n} := \bigoplus_{i \geqslant n} V_i \in \operatorname{GrMod} k$ and the shift $V(n) \in \operatorname{GrMod} k$ by $V(n)_i := V_{n+i}$ for $i \in \mathbb{Z}$. We say that V is right bounded if $V_{\geqslant n} = 0$ for some $n \in \mathbb{Z}$. We say that V is locally finite if $\dim_k V_i < \infty$ for all $i \in \mathbb{Z}$. In this case, we define the Hilbert series of V by

$$H_V(t) := \sum_{i \in \mathbb{Z}} (\dim_k V_i) t^i \in \mathbb{Z}[[t, t^{-1}]].$$