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Automorphisms of polynomial algebras and Dirichlet series

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ABSTRACT

Let $\mathbb{F}_q[x, y]$ be the polynomial algebra in two variables over the finite field $\mathbb{F}_q$ with q elements. We give an exact formula and the asymptotics for the number p_n of automorphisms (f, g) of $\mathbb{F}_q[x, y]$ such that $\max\{\deg(f), \deg(g)\} = n$. We describe also the Dirichlet series generating function

$$p(s) = \sum_{n\geqslant 1} \frac{p_n}{n^s}.$$

The same results hold for the automorphisms of the free associative algebra $\mathbb{F}_q\langle x, y\rangle$. We have also obtained analogues for free algebras with two generators in Nielsen–Schreier varieties of algebras.

Introduction

Our paper is devoted to the following problem. *Let $\mathbb{F}_q[x, y]$ be the polynomial algebra in two variables over the finite field $\mathbb{F}_q$ with q elements. We would like to determine the number of $\mathbb{F}_q$-automorphisms $\varphi = (f, g)$ of $\mathbb{F}_q[x, y]$ satisfying* $\deg(\varphi) := \max\{\deg(f), \deg(g)\} = n$. Here $\varphi = (f, g)$ means that $f = \varphi(x)$, $g = \varphi(y)$.

Our consideration is motivated by Arnaud Bodin [B], who raised the question to determine the number of $\mathbb{F}_q$-automorphisms φ with $\deg(\varphi) \leqslant n$.

In the sequel all automorphisms are $\mathbb{F}_q$-automorphisms. The theorem of Jung and van der Kulk [J,K] states that the automorphisms of the polynomial algebra $K[x, y]$ over any field K are tame. In other words the group $\mathrm{Aut}(K[x, y])$ is generated by the subgroup A of affine automorphisms

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$$\alpha = (a_1x + b_1y + c_1, a_2x + b_2y + c_2), \quad a_i, b_i, c_i \in K,\ a_1b_2 \neq a_2b_1,$$

and the subgroup B of triangular automorphisms

$$\beta = \bigl(ax + h(y), by + b_1\bigr), \quad 0 \neq a, b \in K,\ b_1 \in K,\ h(y) \in K[y].$$

The proof of van der Kulk [K] gives that $\mathrm{Aut}(K[x, y])$ has the following nice structure, see e.g. [C]:

$$\mathrm{Aut}\bigl(K[x, y]\bigr) = A *_C B, \quad C = A \cap B,$$

where $A *_C B$ is the free product of A and B with amalgamated subgroup $C = A \cap B$. Using the canonical form of the elements of $\mathrm{Aut}(K[x, y])$ we have calculated explicitly the number p_n of automorphisms of degree n:

$$p_1 = q^3(q-1)^2(q+1),$$
$$p_n = \bigl(q(q-1)(q+1)\bigr)^2 \sum \left(\frac{q-1}{q}\right)^k q^{n_1+\cdots+n_k}, \quad n > 1,$$

where the summation is on all ordered factorizations $n = n_1 \cdots n_k$ of n, with $n_1, \ldots, n_k > 1$.

It is natural to express the sequence p_n, $n = 1, 2, \ldots,$ in terms of its generating function. When the elements p_n of the sequence involve sums on the divisors of the index n it is convenient to work with the Dirichlet series generating function, i.e., with the formal series

$$p(s) = \sum_{n \geqslant 1} \frac{p_n}{n^s}.$$

For the Riemann zeta function $\zeta(s)$ the coefficients of $\zeta^k(s)$ count the number of ordered factorizations $n = n_1 \cdots n_k$ in k factors. (We have to take $(\zeta(s) - 1)^k$ if we want to count only factorizations with $n_i \geqslant 2$.) Similarly, the coefficients of the kth power $\rho^k(s)$ of the formal Dirichlet series

$$\rho(s) = \sum_{n \geqslant 2} \frac{q^n}{n^s}$$

are equal to $q^{n_1+\cdots+n_k}$ in the expression of p_n. Hence $\rho(s)$ may be considered as a q-analogue of $\zeta(s)$, although it does not satisfy many of the nice properties of the Riemann zeta function (because the sequence q^n is not multiplicative) and for $q > 1$ is not convergent for any nonzero s (because its coefficients grow faster than n^s). We have found that

$$p(s) = \bigl(q(q-1)(q+1)\bigr)^2 \left(\sum_{k \geqslant 0} \frac{q-1}{q} \rho^k(s) - \frac{1}{q+1}\right)$$
$$= \bigl(q(q-1)(q+1)\bigr)^2 \left(\frac{1}{1 - \frac{q-1}{q}\rho(s)} - \frac{1}{q+1}\right)$$

and have given an estimate for the growth of p_n. For $n \geqslant 2$

$$(q-1)^3(q+1)^2q^{n+1} \leqslant p_n \leqslant (q-1)^3(q+1)^2q^{n+1} + (\log_2 n)^{\log_2 n} q^{n/2+8}.$$

Hence for a fixed q and any $\varepsilon > 0$,

$$p_n = (q-1)^3(q+1)^2q^{n+1} + \mathcal{O}\bigl(q^{n(1/2+\varepsilon)}\bigr).$$