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The number of simple modules for the Hecke algebras of type $G(r, p, n)$ (with an appendix by Xiaoyi Cui)

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Abstract

We derive a parameterization of simple modules for the cyclotomic Hecke algebras of type $G(r, p, n)$ with $p > 1$ and $n \geqslant 3$ over fields of any characteristic coprime to p. We give explicit formulas for the number of simple modules over these cyclotomic Hecke algebras.

Keywords: Cyclotomic Hecke algebra; Fock space; Crystal base; Kleshchev r-multipartition; FLOTW r-multipartitions

1. Introduction

Let r, p, d and n be positive integers such that $pd = r$. Let K be a field such that K contains a primitive pth root of unity ε. Let $x_1, \dots, x_d$ be invertible elements in K. Let $q \neq 1$ be an invertible element in K. *Throughout we assume that* $n \geqslant 3$. Let $\mathcal{H}_K(r, n)$ be the unital K-algebra with generators $T_0, T_1, \dots, T_{n-1}$ and relations

$$\begin{aligned}&\left(T_0^p - x_1^p\right)\left(T_0^p - x_2^p\right) \cdots \left(T_0^p - x_d^p\right) = 0,\\&T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0,\end{aligned}$$

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$$(T_i+1)(T_i-q)=0, \quad \text{for } 1\leqslant i\leqslant n-1,$$

$$T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}, \quad \text{for } 1\leqslant i\leqslant n-2,$$

$$T_iT_j=T_jT_i, \quad \text{for } 0\leqslant i<j-1\leqslant n-2.$$

Let $\mathcal{H}_K(r,p,n)$ be the subalgebra of $\mathcal{H}_K(r,n)$ generated by the elements T_0^p, $T_u:=T_0^{-1}T_1T_0$, $T_1,T_2,\dots,T_{n-1}$. This algebra is called the cyclotomic Hecke algebra of type $G(r,p,n)$, which was introduced in [3,6,9]. It includes Hecke algebras of type A, type B and type D as special cases. Note that our assumption $n\geqslant 3$ ensures that we exclude the $G(2r,2p,2)$ case. The generic cyclotomic Hecke algebra of type $G(2r,2p,2)$ has one additional exceptional parameter, thus cannot be realized as a subalgebra of $\mathcal{H}_K(2r,2)$, see [29, §4.B]. The algebra $\mathcal{H}_K(r,1,n)$ is called the Ariki–Koike algebra. These algebras are conjecturely related to Lusztig's induced characters in the modular representation of finite reductive groups over field of nondefining characteristic (see [9]).

The representation of Ariki–Koike algebras (e.g., $\mathcal{H}_K(r,n)$) is well understood by the work of [1,2,11,12]. Let $\mathcal{P}_n$ be the set of r-multipartitions of n. Let $\vec{\mathrm{Q}}:=(Q_1,\dots,Q_r)$ be a fixed arbitrary permutation of

$$\bigl(\underbrace{x_1,x_1\varepsilon,\dots,x_1\varepsilon^{p-1}}_{p\text{ terms}},\dots,\dots,\underbrace{x_d,x_d\varepsilon,\dots,x_d\varepsilon^{p-1}}_{p\text{ terms}}\bigr).$$

We use Q to denote the underlying unordered multiset (allowing repetitions) of $\vec{\mathrm{Q}}$. For any $\lambda\in\mathcal{P}_n$, let $\widetilde{S}^{\lambda}_{\vec{\mathrm{Q}}}$ be the Specht module defined in [11]. There is a naturally defined bilinear form $\langle\,,\rangle$ on $\widetilde{S}^{\lambda}_{\vec{\mathrm{Q}}}$. Let $\widetilde{D}^{\lambda}_{\vec{\mathrm{Q}}}=\widetilde{S}^{\lambda}_{\vec{\mathrm{Q}}}/\operatorname{rad}\langle\,,\rangle$. By [11], the set $\{\widetilde{D}^{\lambda}_{\vec{\mathrm{Q}}}\mid\lambda\in\mathcal{P}_n,\ \widetilde{D}^{\lambda}_{\vec{\mathrm{Q}}}\neq 0\}$ forms a complete set of pairwise nonisomorphic simple $\mathcal{H}_K(r,n)$-modules. By [2] and [12], $\widetilde{D}^{\lambda}_{\vec{\mathrm{Q}}}\neq 0$ if and only if λ is a Kleshchev r-multipartition of n with respect to $(q,\vec{\mathrm{Q}})$.

When $q\neq 1$ is a root of unity, Jacon gives in [26] another parameterization of simple $\mathcal{H}_K(r,n)$-modules via FLOTW r-multipartitions. As an application, a parameterization of simple $\mathcal{H}_K(r,p,n)$-modules is obtained in [17]. The parameterization results in both [17] and [26] are valid only when $K=\mathbb{C}$ (the complex number field). In [21] and [23], using a different approach, we obtain a parameterization of simple $\mathcal{H}_K(p,p,n)$-modules which is valid over field of any characteristic coprime to p, and we give explicit formula for the number of simple modules of $\mathcal{H}_K(p,p,n)$. In this paper, combining the results in [17] with the results and ideas in [23], we derive a parameterization as well as explicit formula for the number of simple $\mathcal{H}_K(r,p,n)$-modules which is valid over fields of any characteristic coprime to p. These results generalize the earlier results in [15,17,19–23,33], and were already announced in [24]. At the end of this paper there is an appendix given by Xiaoyi Cui who fixes a gap in the proof of [16, (2.2)]. We remark that the latter result is crucial to both the present paper and the paper [17].

Throughout this paper, $q\neq 1$ is an invertible element in K. Let e be the smallest positive integer such that $1+q+q^2+\cdots+q^{e-1}=0$ in K; or ∞ if no such positive integer exists. We fix elements $z_1,\dots,z_s\in K^{\times}$, such that $z_iz_j^{-1}\notin q^{\mathbb{Z}}$, $\forall i\neq j$, and for each $1\leqslant i\leqslant r$, $Q_i\in z_jq^{\mathbb{Z}}$ for some $1\leqslant j\leqslant s$.