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Combinatorics in affine flag varieties

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Abstract

The Littelmann path model gives a realization of the crystals of integrable representations of symmetrizable Kac–Moody Lie algebras. Recent work of Gaussent and Littelmann [S. Gaussent, P. Littelmann, LS galleries, the path model, and MV cycles, Duke Math. J. 127 (1) (2005) 35–88] and others [A. Braverman, D. Gaitsgory, Crystals via the affine Grassmannian, Duke Math. J. 107 (3) (2001) 561–575; S. Gaussent, G. Rousseau, Kac–Moody groups, hovels and Littelmann's paths, preprint, arXiv: math.GR/0703639, 2007] has demonstrated a connection between this model and the geometry of the loop Grassmanian. The alcove walk model is a version of the path model which is intimately connected to the combinatorics of the affine Hecke algebra. In this paper we define a refined alcove walk model which encodes the points of the affine flag variety. We show that this combinatorial indexing naturally indexes the cells in generalized Mirković–Vilonen intersections.

Keywords: Loop Grassmannian; Path model; MV cycles

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1. Introduction

A *Chevalley group* is a group in which row reduction works. This means that it is a group with a special set of generators (the "elementary matrices") and relations which are generalizations of the usual row reduction operations. One way to efficiently encode these generators and relations is with a Kac–Moody Lie algebra $\mathfrak{g}$. From the data of the Kac–Moody Lie algebra and a choice of a commutative ring or field $\mathbb{F}$ the group $G(\mathbb{F})$ is built by generators and relations following Chevalley–Steinberg–Tits.

Of particular interest is the case where $\mathbb{F}$ is the field of fractions of $\mathfrak{o}$, the discrete valuation ring $\mathfrak{o}$ is the ring of integers in $\mathbb{F}$, $\mathfrak{p}$ is the unique maximal ideal in $\mathfrak{o}$ and $k = \mathfrak{o}/\mathfrak{p}$ is the residue field. The favorite examples are

$$\begin{array}{lll} \mathbb{F} = \mathbb{C}((t)), & \mathfrak{o} = \mathbb{C}[[t]], & k = \mathbb{C}, \\ \mathbb{F} = \mathbb{Q}_p, & \mathfrak{o} = \mathbb{Z}_p, & k = \mathbb{F}_p, \\ \mathbb{F} = \mathbb{F}_q((t)), & \mathfrak{o} = \mathbb{F}_q[[t]], & k = \mathbb{F}_q, \end{array}$$

where $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Z}_p$ is the ring of p-adic integers, and $\mathbb{F}_q$ is the finite field with q elements. For clarity of presentation we shall work in the first case where $\mathbb{F} = \mathbb{C}((t))$. The diagram

$$\begin{array}{lll} \mathbb{F} & & \\ \cup\!\! \mid & & \\ \mathfrak{o} & \xrightarrow{\mathrm{ev}_{t=0}} & k = \mathfrak{o}/\mathfrak{p} \end{array} \quad \text{gives} \quad \begin{array}{ccccc} G & = & G(\mathbb{C}((t))) & & \\ \cup\!\! \mid & & \cup\!\! \mid & & \\ K & = & G(\mathbb{C}[[t]]) & \xrightarrow{\mathrm{ev}_{t=0}} & G(\mathbb{C}) \\ \cup\!\! \mid & & \cup\!\! \mid & & \cup\!\! \mid \\ I & = & \mathrm{ev}_{t=0}^{-1}(B(\mathbb{C})) & \xrightarrow{\mathrm{ev}_{t=0}} & B(\mathbb{C}) \end{array} \tag{1.1}$$

where $B(\mathbb{C})$ is the "Borel subgroup" of "upper triangular matrices" in $G(\mathbb{C})$. The *loop group* is $G = G(\mathbb{C}((t)))$, I is the standard *Iwahori subgroup* of G,

$$G(\mathbb{C})/B(\mathbb{C}) \text{ is the } \textit{flag variety},$$

$$G/I \text{ is the } \textit{affine flag variety}, \quad \text{and} \quad G/K \text{ is the } \textit{loop Grassmanian}. \tag{1.2}$$

The primary tool for the study of these varieties (ind-schemes) are the following "classical" double coset decompositions, see [St, Ch. 8] and [Mac1, §(2.6)].

Theorem 1.1. *Let W be the Weyl group of $G(\mathbb{C})$, $\widetilde{W} = W \ltimes \mathfrak{h}_{\mathbb{Z}}$ the affine Weyl group, and U^- the subgroup of "unipotent lower triangular" matrices in $G(\mathbb{F})$ and $\mathfrak{h}_{\mathbb{Z}}^+$ the set of dominant elements of $\mathfrak{h}_{\mathbb{Z}}$. Then*

Bruhat decomposition	$G = \bigsqcup_{w \in W} BwB,$	$K = \bigsqcup_{w \in W} IwI,$	
Iwahori decomposition	$G = \bigsqcup_{w \in \widetilde{W}} IwI,$	$G = \bigsqcup_{v \in \widetilde{W}} U^- vI,$	
Cartan decomposition	$G = \bigsqcup_{\lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}^+} K t_{\lambda^\vee} K,$	$G = \bigsqcup_{\mu^\vee \in \mathfrak{h}_{\mathbb{Z}}} U^- t_{\mu^\vee} K,$	*Iwasawa decomposition.*