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Governing singularities of Schubert varieties

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Abstract

We present a combinatorial and computational commutative algebra methodology for studying singularities of Schubert varieties of flag manifolds.

We define the combinatorial notion of *interval pattern avoidance*. For "reasonable" invariants $\mathcal{P}$ of singularities, we geometrically prove that this governs (1) the $\mathcal{P}$-locus of a Schubert variety, and (2) which Schubert varieties are globally not $\mathcal{P}$. The prototypical case is $\mathcal{P} =$ "singular"; *classical* pattern avoidance applies admirably for this choice [V. Lakshmibai, B. Sandhya, Criterion for smoothness of Schubert varieties in $\mathrm{SL}(n)/B$, Proc. Indian Acad. Sci. Math. Sci. 100 (1) (1990) 45–52, MR 91c:14061], but is insufficient in general.

Our approach is analyzed for some common invariants, including Kazhdan–Lusztig polynomials, multiplicity, factoriality, and Gorensteinness, extending [A. Woo, A. Yong, When is a Schubert variety Gorenstein?, Adv. Math. 207 (1) (2006) 205–220, MR 2264071]; the description of the singular locus (which was independently proved by [S. Billey, G. Warrington, Maximal singular loci of Schubert varieties in $\mathrm{SL}(n)/B$, Trans. Amer. Math. Soc. 335 (2003) 3915–3945, MR 2004f:14071; A. Cortez, Singularités génériques et quasi-résolutions des variétés de Schubert pour le groupe linéaire, Adv. Math. 178 (2003) 396–445, MR 2004i:14056; C. Kassel, A. Lascoux, C. Reutenauer, The singular locus of a Schubert variety, J. Algebra 269 (2003) 74–108, MR 2005f:14096; L. Manivel, Le lieu singulier des variétés de Schubert, Int. Math. Res. Not. 16 (2001) 849–871, MR 2002i:14045]) is also thus reinterpreted.

Our methods are amenable to computer experimentation, based on computing with *Kazhdan–Lusztig ideals* (a class of generalized determinantal ideals) using `Macaulay 2`. This feature is supplemented by a collection of open problems and conjectures.

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Contents

1. Overview

Let X_w be the Schubert variety of the complete flag variety Flags($\mathbb{C}^n$) associated to a permutation w in the symmetric group S_n. One would like to describe and classify the singularities of X_w, as well as calculate invariants measuring their complexity. Solutions to such problems typically require techniques from and have important applications to geometry, representation theory, and associated combinatorics. Two recent surveys of some work in this area are [2,9].

In this paper, we formulate a new combinatorial notion, a generalization of pattern avoidance we call *interval pattern avoidance*; we then use this idea to explore the singularities of Schubert varieties and their local invariants. The well-known Kazhdan–Lusztig polynomials show up as one local invariant, since their coefficients are the Betti numbers for the local intersection cohomology of the singularities. Indeed, a desire to further understand the combinatorics of Kazhdan–Lusztig polynomials is one source of motivation (and application) for this present work. However, there are many other noteworthy invariants of singularities, including factoriality, multiplicity, Gorensteinness, and Cohen–Macaulay type. We provide a uniform language to study such *semicontinuously stable* invariants, in an attempt to gain further insight into the singularities of Schubert varieties.

Informally, our principal thesis is that, for any of these "reasonable" local invariants of singularities of Schubert varieties, the question of where it assumes a particular value has a natural answer in terms of interval pattern avoidance. Our main result (Theorem 2.6) is a precise version of this assertion, together with a geometric explanation; proofs are given in Section 4.

The two most basic problems about singularities of specific Schubert varieties are