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ScienceDirect
Journal of Algebra 319 (2008) 3567–3587
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Finite index supergroups and subgroups of torsionfree abelian groups of rank two

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Received 8 June 2004
Available online 4 March 2008
Communicated by Michel Broué

Abstract

Every torsionfree abelian group A of rank two is a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ and is expressed by a direct limit of free abelian groups of rank two with lower diagonal integer-valued 2×2-matrices as the bonding maps. Using these direct systems we classify all subgroups of $\mathbb{Q} \oplus \mathbb{Q}$ which are finite index supergroups of A or finite index subgroups of A. Using this classification we prove that for each prime p there exists a torsionfree abelian group A satisfying the following, where $A \leqslant \mathbb{Q} \oplus \mathbb{Q}$ and all supergroups are subgroups of $\mathbb{Q} \oplus \mathbb{Q}$:

(1) for each natural number s there are $\sum_{q|s,\, \gcd(p,q)=1} q$ s-index supergroups and also $\sum_{q|s,\, \gcd(p,q)=1} q$ s-index subgroups;
(2) each pair of distinct s-index supergroups are non-isomorphic and each pair of distinct s-index subgroups are non-isomorphic.

Keywords: Torsionfree abelian group; Rank two; Finite index; Subgroup; Supergroup

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doi:10.1016/j.jalgebra.2007.08.033

1. Introduction and main results

This investigation originally started from a classification of finite-sheeted covering maps on connected compact abelian groups. When groups are 1-dimensional, a classification is fairly easy [2], which is reduced in principle to Baer's classification of torsionfree abelian groups of rank one. As a next step we have investigated the 2-dimensional case, which will appear in another paper [4]. In that paper we proved the following for a connected compact group Y:

(a) Every finite-sheeted covering map from a connected space over Y is equivalent to a covering homomorphism from a compact, connected group. Moreover, if Y is abelian, then the domain of the homomorphism is abelian.
(b) Let $f: X \to Y$ and $f': X' \to Y$ be finite-sheeted covering homomorphisms over Y. Then f and f' are equivalent as covering maps if and only if the two homomorphisms are equivalent as topological homomorphisms.

Accordingly we can reduce all things to the category of compact abelian groups, and then, by the Pontrjagin duality, it reduces further to an investigation of the equivalence class of finite index supergroups of torsionfree abelian groups of rank two. Here, two supergroups B and C of a group A are equivalent, if there exists an isomorphism between B and C which fixes every element of A. When B and C are finite index supergroups of A, the embedding of A to the direct sum of two copies of the rational group $\mathbb{Q} \oplus \mathbb{Q}$ induces embeddings of B and C to $\mathbb{Q} \oplus \mathbb{Q}$ and then equivalent supergroups B and C are mapped onto the same subgroup of $\mathbb{Q} \oplus \mathbb{Q}$. From now on, when we consider a supergroup of a torsionfree abelian group A of rank two, we assume that A is embedded into $\mathbb{Q} \oplus \mathbb{Q}$ and the supergroup is a subgroup of $\mathbb{Q} \oplus \mathbb{Q}$.

Every torsionfree abelian group A of rank two is presented by $A = \varinjlim(A_n, g_n\colon n < \omega)$ where A_n's are copies of $\mathbb{Z} \oplus \mathbb{Z}$ and $g_n = \left[\begin{smallmatrix} p_n & 0 \\ \alpha_n & t_n \end{smallmatrix}\right] \in M_2(\mathbb{Z})$ such that $p_n, t_n > 0$ and $0 \leqslant \alpha_n < p_n$. For a natural number s let F_s be the set of all positive integers q satisfying $\gcd(p_n, q) = 1$ for almost all n and that there exists q_1 such that $qq_1r = s$ and

(a) $\gcd(p_n, q_1) = \gcd(t_n, r) = 1$ for almost all n;
(b) if $q_1 > 1$, the $\gcd(t_n, q_1) \neq 1$ for infinitely many n's.

Under the above presentation of A we prove the following:

(1) For a natural number s, the number of distinct s-index supergroups of A is $\sum_{q \in F_s} q$ and the number of s-index subgroups of A is also $\sum_{q \in F_s} q$.
(2) Let $(\alpha_n\colon n < \omega)$ be semi-periodic and p a positive integer. If $p_n = p$, $t_n = 1$ for almost all n or if $p_n = t_n = p$ for almost all n, then finite index supergroups of A are isomorphic to A, and all finite index subgroups of A are also isomorphic to A (Corollary 6.7).
(3) Let p be a prime and $p_n = p$, $t_n = 1$ for every n and q be a natural number with $q > 1$ and $\gcd(p, q) = 1$. Let a p-adic integer $\sum_{n=0}^{\infty} \alpha_n p^n$ is not quadratic over $\mathbb{Q}$. Then, for each natural number s, distinct s-index supergroups of A are non-isomorphic. Moreover, distinct s-index subgroups of A are non-isomorphic (Theorem 5.2).

A restricted form of (3) was asserted in our former paper [3].

In the second section of the present paper we explain how to express a rank 2 torsionfree abelian group, its finite index supergroups, and its finite index subgroups by a sequence of