Journal of Algebra 320 (2008) 4102–4131

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Infinitesimal deformations of restricted simple Lie algebras I

Filippo Viviani [1]

Institut für Mathematik, Humboldt Universität zu Berlin, 10099 Berlin, Germany

ARTICLE INFO

Article history:
Received 3 January 2007
Available online 11 October 2008
Communicated by Vera Serganova

Keywords:
Deformations
Restricted simple Lie algebras of Cartan type

ABSTRACT

We compute the infinitesimal deformations of two families of restricted simple modular Lie algebras of Cartan-type: the Witt–Jacobson and the Special Lie algebras.

1. Introduction

Simple Lie algebras over an algebraically closed field of characteristic zero were classified at the beginning of the XIX century by Killing and Cartan. They used the non-degeneracy of the Killing form to describe the simple Lie algebras in terms of root systems which are then classified by Dynkin diagrams.

This method breaks down in positive characteristic because the Killing form may degenerate. Indeed the classification problem remained open for a long time until it was recently solved, if the characteristic of the base field is greater than 3, by Block and Wilson (see [BW88]), Strade and Wilson (see [SW91]), Strade (see [STR89,STR92,STR91,STR93,STR94,STR98]) and Premet and Strade (see [PS97, PS99,PS01]). The classification remains still open in characteristic 2 and 3 (see [STR04, p. 209]).

According to this classification, *simple modular* (that is over a field of positive characteristic) Lie algebras are divided into two big families, called classical-type and Cartan-type algebras. The algebras of classical-type are obtained by the simple Lie algebras in characteristic zero by first taking a model over the integers (via Chevalley bases) and then reducing modulo p (see [SEL67]). The algebras of Cartan-type were constructed by Kostrikin and Shafarevich in 1966 (see [KS66]) as finite-dimensional analogues of the infinite-dimensional complex simple Lie algebras, which occurred in Cartan's classification of Lie pseudogroups, and are divided into four families, called Witt–Jacobson,

E-mail address: viviani@math.hu-berlin.de.

[1] During the preparation of this paper, the author was partially supported by a grant from the Mittag-Leffler Institute in Stockholm.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.08.022

Special, Hamiltonian and Contact algebras. The Witt–Jacobson Lie algebras are derivation algebras of truncated divided power algebras and the remaining three families are the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. Moreover in characteristic 5 there is one exceptional simple modular Lie algebra called the Melikian algebra (introduced in [MEL80]).

We are interested in a particular class of modular Lie algebras called *restricted*. These can be characterized as those modular Lie algebras such that the p-power of an inner derivation (which in characteristic p is a derivation) is still inner. Important examples of restricted Lie algebras are the ones coming from groups schemes. Indeed there is a one-to-one correspondence between restricted Lie algebras and finite group schemes whose Frobenius vanishes (see [DG70, Chapter 2]).

By standard facts of deformation theory, the *infinitesimal deformations* of a Lie algebra are parametrized by the second cohomology of the Lie algebra with values in the adjoint representation (see for example [GER64]).

It is a classical result (see [HS97]) that for a simple Lie algebra $\mathfrak{g}$ over a field of characteristic 0 it holds that $H^i(\mathfrak{g},\mathfrak{g})=0$ for every $i\geqslant 0$, which implies in particular that such Lie algebras are rigid. The proof of this fact relies on the non-degeneracy of the Killing form and the non-vanishing of the trace of the Casimir element, which is equal to the dimension of the Lie algebra. Therefore the same proof works also for the simple modular Lie algebras of classical type over a field of characteristic not dividing the determinant of the Killing form and the dimension of the Lie algebra. Actually Rudakov (see [RUD71]) showed that such Lie algebras are rigid if the characteristic of the base field is greater than or equal to 5 while in characteristic 2 and 3 there are non-rigid classical Lie algebras (see [CHE05,CK00,CKK00]).

The purpose of this article is to compute the infinitesimal deformations of the first two families of restricted simple Lie algebras of Cartan type: the Witt–Jacobson algebras $W(n)$ and the Special algebras $S(n)$. Unlike the classical-type simple algebras, it turns out that these two families are not rigid. More precisely we get the following two theorems (we refer to Sections 3.1 and 4.1 for the standard notations concerning $W(n)$ and $S(n)$ and to Section 2.3 for the definition of the squaring operators Sq).

Theorem 1.1. *Assume that the characteristic p of the base field F is different from 2. Then we have*

$$H^2\big(W(n),W(n)\big)=\bigoplus_{i=1}^{n}F\cdot\big\langle \mathrm{Sq}(D_i)\big\rangle$$

with the exception of the case $n=1$ and $p=3$ when it is 0.

Theorem 1.2. *Assume that the characteristic of the base field F is different from 2 and moreover it is different from 3 if $n=3$. Then we have*

$$H^2\big(S(n),S(n)\big)=\bigoplus_{i=1}^{n}F\cdot\big\langle \mathrm{Sq}(D_i)\big\rangle\bigoplus F\cdot\langle\Theta\rangle$$

where Θ is defined by $\Theta(D_i,D_j)=D_{ij}(x^{\tau})$ and extended by 0 outside $S(n)_{-1}\times S(n)_{-1}$.

In the two forthcoming papers [VIV2,VIV3], we compute the infinitesimal deformations of the remaining restricted simple Lie algebras of Cartan-type, namely the Hamiltonian, the Contact and the exceptional Melikian algebras. Moreover, in the forthcoming paper [VIV4], we apply these results to the study of the infinitesimal deformations of the simple finite group schemes corresponding to the simple restricted Lie algebras of Cartan type.

Let us mention that the infinitesimal deformations of simple Lie algebras of Cartan-type (in the general non-restricted case) have been considered already by Džumadildaev in [DZU80,DZU81, DZU89] and Džumadildaev and Kostrikin in [DK78] but a complete picture as well as detailed proofs