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Whittaker modules for Heisenberg algebras and imaginary Whittaker modules for affine Lie algebras

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ABSTRACT

We classify the irreducible Whittaker modules for finite- and infinite-dimensional Heisenberg algebras and for the Lie algebra obtained by adjoining a degree derivation to an infinite-dimensional Heisenberg algebra. We use these modules to construct a new class of modules for non-twisted affine Lie algebras, which we call imaginary Whittaker modules. We show that the imaginary Whittaker modules of non-zero level are irreducible.

1. Introduction

In R. Block's classification [3] of all irreducible modules for the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 complex matrices, the irreducible $\mathfrak{sl}_2$-modules fall into three families: highest (lowest) weight modules, *Whittaker* modules, and a third family obtained by localization. This result illustrates the prominent role played by Whittaker modules. The class of Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ was defined by B. Kostant in [17]. Kostant showed that these modules are (up to isomorphism) in bijective correspondence with ideals of the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. In particular, irreducible Whittaker modules correspond to maximal ideals of $\mathcal{Z}(\mathfrak{g})$. In [22], N. Wallach gave new proofs of Kostant's results in the case $\mathfrak{g}$ is the product of complex Lie algebras isomorphic to $\mathfrak{sl}_n$.

In the quantum setting, M. Ondrus classified Whittaker modules for the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ in [20], and studied their tensor products with finite-dimensional modules for

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$\mathcal{U}_q(\mathfrak{sl}_2)$ in [21]. Recently, G. Benkart and M. Ondrus investigated Whittaker modules for generalized Weyl algebras in [2].

E. McDowell [18], and D. Miličić and W. Soergel [19] studied a category of modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ which includes the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ as well as those Whittaker modules W which are locally finite over the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ (i.e. $\mathcal{Z}(\mathfrak{g})v$ is finite-dimensional for each $v \in W$). The irreducible objects in this category are constructed by inducing over a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ from an irreducible Whittaker module (in Kostant's sense) or from a highest weight module for the reductive Levi factor of $\mathfrak{p}$.

In this paper we construct Whittaker type modules for non-twisted affine Lie algebras using parabolic induction. The parabolic subalgebras for affine Lie algebras were classified by V. Futorny in [11]. In particular, they fall into two types depending on their Levi factor, which is either a finite-dimensional reductive Lie algebra or the sum of an infinite-dimensional Heisenberg Lie algebra with a Cartan subalgebra and possibly with the derived algebras of affine Lie subalgebras of smaller rank. *Loop modules* for an affine Lie algebra $\mathfrak{g}$ are modules induced over a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l} = \mathfrak{t} + \mathfrak{h}$ from irreducible $\mathbb{Z}$-graded $\mathfrak{t}$-modules, where $\mathfrak{t}$ is an infinite-dimensional Heisenberg subalgebra, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The central element of $\mathfrak{g}$ then acts as a scalar called the level of the module. Integrable loop modules of zero level were studied in [5–8], but loop modules of zero level are still not completely classified. Loop modules of non-zero level are also called imaginary Verma modules, and they were studied in [12]. Analogues of imaginary Verma modules have also been constructed for the quantum group $\mathcal{U}_q(\mathfrak{g})$ of a non-twisted affine Lie algebra $\mathfrak{g}$ in [14] and for the extended affine Lie algebra $\mathfrak{sl}_2(\mathbb{C}_q)$ in [10].

Let $\mathfrak{t} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{t}_i$ be a Heisenberg Lie algebra with a one-dimensional center $\mathfrak{t}_0 = \mathbb{C}c$. The Heisenberg Lie algebras we consider in this work are infinite-dimensional and of a particular type, as they are the homogeneous Heisenberg subalgebras of affine Lie algebras. If V is an irreducible $\mathfrak{t}$-module, then c acts as a scalar, called the *level*. In this paper, we describe the irreducible Whittaker modules for a Heisenberg Lie algebra $\mathfrak{t}$. These modules are not $\mathbb{Z}$-graded as $\mathfrak{t}$-modules. All our results are also valid for any finite-dimensional Heisenberg Lie algebra with some minor modifications in the definitions. From the Whittaker $\mathfrak{t}$-modules of level one, we obtain irreducible Whittaker modules for Weyl algebras. The irreducible $\mathbb{Z}$-graded $\mathfrak{t}$-modules $V = \bigoplus_{i\in\mathbb{Z}} V_i$ with non-zero level and $0 < \dim_{\mathbb{C}} V_i < \infty$ for at least one i have been described in [13]. Examples of irreducible $\mathbb{Z}$-graded $\mathfrak{t}$-modules with a non-zero level and $\dim_{\mathbb{C}} V_i = \infty$ for all i were constructed in [1]. In [5], V. Chari classified the irreducible $\mathbb{Z}$-graded $\mathfrak{t}$-modules of zero level. The modules we study are different from all these as they are not $\mathbb{Z}$-graded as $\mathfrak{t}$-modules. We also classify the irreducible Whittaker modules for the Lie algebra $\tilde{\mathfrak{t}}$ obtained by adjoining a degree derivation d to $\mathfrak{t}$. We construct a new class of modules for non-twisted affine Lie algebras from irreducible Whittaker modules for the Lie algebra $\tilde{\mathfrak{t}}$. We call these modules *imaginary Whittaker modules*, as they are constructed by inducing over the same parabolic subalgebra as imaginary Verma modules or loop modules, but with the root vectors corresponding to the imaginary roots acting in a non-zero fashion. We prove that the imaginary Whittaker modules of non-zero level are irreducible. In [9], we have also studied Whittaker type modules induced from parabolic subalgebras with a finite-dimensional reductive Levi factor, and we have classified all irreducible Whittaker modules for $\widehat{\mathfrak{sl}_2}$ satisfying certain assumptions.

Here is a brief outline of the paper. In Section 2, we recall background information and establish notation. In Section 3, we determine all the irreducible Whittaker modules for Heisenberg Lie algebras. We study the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{t})$ of a Heisenberg Lie algebra $\mathfrak{t}$ and the annihilator ideals of irreducible Whittaker $\mathfrak{t}$-modules of non-zero level in Section 4. In Section 5, we describe the irreducible Whittaker modules for the Lie algebra $\tilde{\mathfrak{t}}$ obtained by adjoining a derivation to $\mathfrak{t}$. Finally, in the last section we construct imaginary Whittaker modules for a non-twisted affine Lie algebra and show that the imaginary Whittaker modules of non-zero level are irreducible.

2. Preliminaries

For any algebra $\mathcal{A}$ (Lie or associative) we denote its center by $\mathcal{Z}(\mathcal{A})$.

Let n be a positive integer and let $\mathfrak{t}$ be a Lie algebra over $\mathbb{C}$ with the following properties:

(i) $\mathfrak{t}$ has a one-dimensional center, $\mathcal{Z}(\mathfrak{t}) = \mathbb{C}c$, and $\mathcal{Z}(\mathfrak{t}) = [\mathfrak{t}, \mathfrak{t}]$,