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ScienceDirect
Journal of Algebra 319 (2008) 1825–1850
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

On uniserial modules in the Auslander–Reiten quiver

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Received 12 April 2004
Available online 11 January 2008
Communicated by Kent R. Fuller

Abstract

This article begins the study of irreducible maps involving finite-dimensional uniserial modules over finite-dimensional associative algebras. We work on the classification of irreducible maps between two uniserials over triangular algebras, and give estimates for the number of middle terms of an almost split sequence with a uniserial end term.

Keywords: Auslander–Reiten quiver; Uniserial modules; Irreducible maps; Almost split sequences

1. Introduction

The study of finite-dimensional uniserial modules over finite-dimensional associative algebras was begun in earnest by Huisgen-Zimmermann in [8]; Huisgen-Zimmermann and Bongartz achieved a description of uniserial modules in terms of certain varieties in [5] (see also [4,6,9]). In the present article, which is based on the authors' theses [3,12], certain questions regarding the position of uniserial modules in the Auslander–Reiten quiver of finite-dimensional algebras are investigated; most of the work applies to basic split triangular algebras only.

The article is organized as follows. In Section 2 we fix our notation and conventions and recall the basic description of uniserials via varieties. In Section 3 we present a general result that motivates much of the following work: any irreducible map between two uniserials is either the radical embedding or the socle factor projection of a uniserial module. The two cases being dual,

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2007.11.026

we go on to state a conjecture giving a concrete necessary and sufficient condition for a uniserial over a triangular algebra to have an irreducible radical embedding. The criterion is combinatorial in nature—as a consequence, while slightly technical when phrased in full generality, it is readily checkable for a given quiver with relations. The sufficiency of this condition is proved using the technique of quiver representations. The necessity of one part of the condition is then proved in a slightly more general context.

We have not yet managed to prove the necessity of the full condition for all triangular algebras. In Section 4 we prove it under an additional assumption, which includes the case of all triangular multiserial algebras. In Section 5 we prove it for all monomial algebras; the condition takes on a very simple form in this situation.

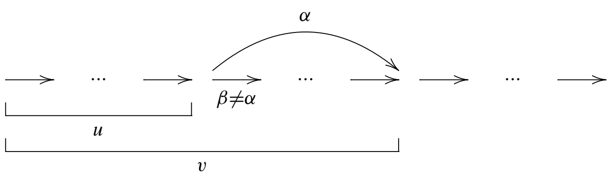
In Section 6 we study a general finite-dimensional algebra and focus on a different circle of questions: almost split sequences with a uniserial end term. First we give a simple general result: any short exact sequence with uniserial end terms has a middle term which is either indecomposable or a direct sum of two uniserials. Then we study the number of indecomposable middle terms in an almost split sequence ending in a uniserial module; an upper bound is given for multiserial algebras.

2. Notation and preliminaries

We will use the notation and terminology of [1]. Throughout, $\mathfrak{K}$ will be a field and Γ will be a finite quiver with vertex set Γ_0 and arrow set Γ_1. We compose arrows, paths and maps from right to left: if $p : e \to f$ and $q : f \to g$ then $qp : e \to g$. The starting point of the path p is denoted by $\mathrm{s}(p)$ and its end point by $\mathrm{t}(p)$. $\Lambda = \mathfrak{K}\Gamma/I$ will be a finite-dimensional $\mathfrak{K}$-algebra presented as the quotient of the path algebra of Γ by an admissible ideal I. Λ is called *triangular* if Γ does not contain any directed cycles. Whenever useful, we identify elements of Γ_0 and paths in Γ with their corresponding classes in Λ.

The category of finitely generated left Λ-modules is denoted by Λ-mod. The direct sum of two modules M and N is denoted by $M \sqcup N$. A module is called *uniserial* if it has only one composition series with simple factors. If $U \in \Lambda$-mod is uniserial with length n, then there exists a path p in Γ of length $n-1$ and an element $x \in U$ such that $px \neq 0$. Any such path is called a *mast* of U and any such element x is called a *top element* of U. The terminology is that of [8].

Let p be a path in Γ. A path u is a *right subpath* of p if there exists a path r with $p = ru$. Following [8], a *detour* on the path p is a pair (α, u) with α an arrow and u a right subpath of p, where αu is a path in Γ which is not a right subpath of p, but there exists a right subpath v of p with $\mathrm{length}(v) \geqslant \mathrm{length}(u) + 1$ such that the endpoint of v coincides with the endpoint of α.



We will abbreviate the statement "(α, u) is a detour on p" by $(\alpha, u) \wr\wr p$. Given any detour on p, let $V(\alpha, u) = \{v_i(\alpha, u) \mid i \in I(\alpha, u)\}$ be the family of right subpaths of p in $\mathfrak{K}\Gamma$ which are longer than u and have the same endpoint as α.

Now suppose p has length l and passes consecutively through the vertices $e(1), \ldots, e(l+1)$ (which need not be distinct). A *route* on p is any path in Γ which starts in $e(1)$ and passes