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# SRAR loops with more than two commutators ☆

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**Abstract**

Possession of a unique nonidentity commutator/associator is a property that dominates the theory of loops whose loop rings, while not associative, nevertheless satisfy an "interesting" identity. For instance, until now, all loops with loop rings satisfying the right Bol identity (such loops are called SRAR) have been known to have this property. In this paper, we present various constructions of other kinds of SRAR loops.


*Keywords:* Bol loop; Left nucleus; Commutator; Strongly right alternative

## 1. Introduction

For any associative, commutative ring $R$ with 1 and any loop $L$, one can construct the loop ring $RL$ precisely as if $L$ were a group. A half century ago, with mild restrictions on characteristic, Lowell Paige proved that if a commutative loop ring is even power associative, then that loop ring and hence the underlying loop as well must be associative [Pai55]. Such observations are perhaps the reason that the loop ring in general remained an almost forgotten object for thirty years. In the mid 1980s, the second author found a class of loops whose loop rings satisfy both the right and left alternative laws in characteristic different from 2 [Goo83]. Since the underlying

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loop $L$ is contained in the loop ring $RL$ and since alternative rings are known to satisfy the three (equivalent) Moufang identities

$$(xy)(zx) = x\big[(yz)x\big],$$
$$\big[(xy)z\big]y = x\big[y(zy)\big],$$
$$x\big[y(xz)\big] = \big[(xy)x\big]z,$$

the loop $L$ must be a Moufang loop. But not all Moufang loops give rise to alternative loop rings. In [CG86], the authors found precise conditions on the Moufang loop $L$ under which the loop ring $RL$, in characteristic different from 2, is alternative. Later, they found more Moufang loops whose loop rings are alternative in any characteristic [CG90] and, in the 1990s, Goodaire and Robinson found some (nonassociative) Bol loops[1] whose loop rings (necessarily in characteristic 2 [Kun98]) are *strongly right alternative* in the sense that they satisfy not just the right alternative law, but the stronger right Bol identity [GR95]. Such loops are called *SRAR*.

Historically, one loop theoretic property has been dominant amongst those classes of loops whose loop rings satisfy an identity of Bol–Moufang type (an identity of degree four in three variables, such as the Moufang or Bol identities). The property in question is the possession of a unique nonidentity commutator/associator, i.e., an element $s$ such that, for all elements $a$ and $b$ in the loop,

$$ab = ba \quad \text{or} \quad ab = (ba)s \tag{1.1}$$

and, for all elements $a$, $b$ and $c$,

$$(ab)c = a(bc) \quad \text{or} \quad (ab)c = \big[a(bc)\big]s. \tag{1.2}$$

For instance, until now, all known SRAR loops have had this property (and all Bol loops with this property are SRAR) and it has been tempting to conjecture that only Bol loops with a unique nonidentity commutator/associator can be SRAR. The purpose of this paper is to show that this conjecture is false.

If $L$ is a loop and if $a$, $b$ and $c$ are elements of $L$, we use $(a,b)$ to denote the commutator of $a$ and $b$ (this is the element $s$ which appears in (1.1)) and $(a,b,c)$ to denote the associator of $a$, $b$ and $c$ (this is the element $s$ which appears in (1.2)). The subloop of $L$ generated by all commutators and associators is denoted $L'$. Thus the assertion that $L$ has a unique nonidentity commutator/associator is the statement $|L'| = 2$.

The *left nucleus* of $L$ is the set

$$N_\lambda = \big\{x \in L \bigm| (x,a,b) = 1 \text{ for all } a, b \in L\big\}.$$

A good reference for the theory of loops and Bol loops in particular is the text by Hala Pflugfelder [Pfl90]. Key properties of Bol loops include their *power associativity* (powers of an element are well-defined) and, more generally, their right *power alternativity*: $(ab^i)b^j = ab^{i+j}$ for all $a$, $b$

[1] In this paper, “nonassociative” always means “not associative” and all Bol loops are right Bol, that is, they satisfy the right Bol identity $[(xy)z]y = x[(yz)y]$.