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Mickelsson algebras and Zhelobenko operators

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Abstract

We construct a family of automorphisms of Mickelsson algebra, satisfying braid group relations. The construction uses 'Zhelobenko cocycle' and includes the dynamical Weyl group action as a particular case.

Keywords: Contragredient Lie algebras of finite growth; Quantized universal enveloping algebras; Extremal projector; Mickelsson algebras; Zhelobenko cocycles; Braid group actions; Dynamical Weyl group

Contents

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1. Introduction

Mickelsson algebras were introduced in [M] for the study of Harish-Chandra modules of reductive groups. The Mickelsson algebra, related to a real reductive group, acts on the space of highest weight vectors of its maximal compact subgroup, and each irreducible Harish-Chandra module of the initial reductive group is uniquely characterized by this action.

A similar construction can be given for any associative algebra $\mathcal{A}$, which contains a universal enveloping algebra $U(\mathfrak{g})$ (or its q-analog) of a contragredient Lie algebra $\mathfrak{g}$ with a fixed Gauss decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$. Namely, we define the Mickelsson algebra $S^{\mathfrak{n}}(\mathcal{A})$ as the quotient of the normalizer $N(\mathcal{A}\mathfrak{n})$ of the left ideal $\mathcal{A}\mathfrak{n}$ by this ideal. For any representation V of $\mathcal{A}$ the Mickelsson algebra $S^{\mathfrak{n}}(\mathcal{A})$ acts on the space $V^{\mathfrak{n}}$ of $\mathfrak{n}$-invariant vectors. This construction provides a reduction of a representation of $\mathcal{A}$ with respect to the action of $U(\mathfrak{g})$ and can be viewed as a counterpart of Hamiltonian reduction. It has been applied for various problems of representation theory, see the survey [T2] and references therein.

The structure of Mickelsson algebra simplifies after localizing it over a certain multiplicative subset of $U(\mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The corresponding algebra $Z^{\mathfrak{n}}(\mathcal{A})$ is generated by a finite-dimensional space of generators, which obey quadratic-linear relations. These generators can be defined with a help of an extremal projector of Asherova–Smirnov–Tolstoy [AST]. An application of the extremal projector to the study of the Mickelsson algebras $Z^{\mathfrak{n}}(\mathcal{A})$ was proposed by Zhelobenko [Z2]. Besides, Zhelobenko developed the so called 'dual methods,' and constructed another set of generators of the Mickelsson algebra by means of a family of special operators, which form a cocycle on the Weyl group [Z1].

Later Mickelsson algebras appeared in the theory of dynamical quantum groups. Their basic ingredients, the intertwining operators between Verma modules and the tensor products of Verma modules with finite-dimensional representations actually form special Mickelsson algebras. Matrix coefficients of these intertwining operators are very useful in the study of quantum integrable models [ES]. Tarasov and Varchenko [TV] found the symmetries of the algebra of intertwining operators, which originate from the morphisms of Verma modules. They satisfy the braid group relations and transform the weights of the Cartan subalgebra by means of a shifted Weyl group action. These symmetries got the name of a 'dynamical Weyl group.' The theory of dynamical Weyl groups was generalized to the quantum groups setup in [EV]. The form of operators of the dynamical Weyl group is very close to the factorized expressions for the extremal projector and for the Zhelobenko cocycles [Z1]. However, the precise statements and the origin of such a relation were not clear. One of our goals is to clarify this relation.

In this paper we describe a family of symmetries for a wide class of Mickelsson algebras. They form a representation of the related braid group by automorphisms of the Mickelsson algebra $Z^{\mathfrak{n}}(\mathcal{A})$ and transform the Cartan elements by means of the shifted Weyl group action. Each generating automorphism is a product of the Zhelobenko 'cocycle' map q_{α_i} and of an automorphism T_i of the algebra $\mathcal{A}$, extending the action of the Weyl group on the Cartan subalgebra. The new feature of our approach is the homomorphism property of Zhelobenko maps, that was not noticed before. However, the proof of this property is not short and requires calculations with the extremal projector.