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Automatic subsemigroups of free products ☆

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Abstract

We consider the automaticity of subsemigroups of free products of semigroups, proving that subsemigroups of free products, with all generators having length greater than one in the free product, are automatic. As a corollary, we show that if S is a free product of semigroups that are either finite or free, then any finitely generated subsemigroup of S is automatic. In particular, any finitely generated subsemigroup of a free product of finite or monogenic semigroups is automatic.

Keywords: Automatic; Semigroup; Free product; Automata

1. Introduction

The notion of automatic group has recently been extended to semigroups and the basic properties of this new class of semigroups have been established in [3]. The notion of automatic semigroup does not correspond to a nice geometric property as in the case of groups where being automatic is the same as having the fellow traveler property (see [1,2]). Nevertheless it is a natural class of semigroups where we have some interesting computational properties, for example, the word problem is solvable in quadratic time (see [3]), and several results concerning automaticity of semigroups have been established (see, for example, [4,7–10]).

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We are interested in the following general question:

When is a subsemigroup of an automatic semigroup automatic as well?

A general result concerning this problem was established in [9], where the authors have proved the following:

Proposition 1.1. *Let S be a semigroup and let T be a subsemigroup of S such that $S\backslash T$ is finite. Then S is automatic if and only if T is automatic.*

A description of the subsemigroups of the bicyclic monoid, the well-known semigroup defined by the presentation $\langle b, c \mid bc = 1\rangle$, was obtained in [5] and, using this description, the question above was answered (in [6]) for the bicyclic monoid and its subsemigroups:

Proposition 1.2. *All finitely generated subsemigroups of the bicyclic monoid are automatic.*

The question was also solved (in [3]) for free semigroups and their subsemigroups where the following was shown:

Proposition 1.3. *If F is a free semigroup and S is a finitely generated subsemigroup of F, then S is automatic.*

In this paper we extend this last result by considering subsemigroups of free products of semigroups. We show that some subsemigroups of free products of arbitrary semigroups, including in particular finitely generated subsemigroups of free semigroups, are automatic.

We start by introducing the definitions we require. Given a finite set A, which we call an *alphabet*, we denote by A^+ the free semigroup generated by A consisting of finite sequences of elements of A, which we call *words*, under the concatenation, and by A^* the free monoid generated by A consisting of A^+ together with the empty word ϵ. Let S be a semigroup and $\psi : A \to S$ a mapping. We say that A is a *finite generating set for S with respect to ψ* if the unique extension of ψ to a semigroup homomorphism $\psi : A^+ \to S$ is surjective. For $u, v \in A^+$ we write $u \equiv v$ to mean that u and v are equal as words and $u = v$ to mean that u and v represent the same element in the semigroup, i.e. that $u\psi = v\psi$. We say that a subset L of A^+, usually called a *language*, is *regular* if there is a finite state automaton accepting L. To be able to deal with automata that accept pairs of words and to define automatic semigroups we need to define the set $A(2,\$) = ((A \cup \{\$\}) \times (A \cup \{\$\}))\backslash\{(\$,\$)\}$ where \$ is a symbol not in A (called the *padding symbol*) and the function $\delta_A : A^* \times A^* \to A(2,\$)^*$ defined by

$$(a_1 \ldots a_m, b_1 \ldots b_n)\delta_A = \begin{cases} \epsilon & \text{if } 0 = m = n, \\ (a_1, b_1) \ldots (a_m, b_m) & \text{if } 0 < m = n, \\ (a_1, b_1) \ldots (a_m, b_m)(\$, b_{m+1}) \ldots (\$, b_n) & \text{if } 0 \leqslant m < n, \\ (a_1, b_1) \ldots (a_n, b_n)(a_{n+1}, \$) \ldots (a_m, \$) & \text{if } m > n \geqslant 0. \end{cases}$$

Let S be a semigroup and A a finite generating set for S with respect to $\psi : A^+ \to S$. The pair (A, L) is an *automatic structure for S (with respect to ψ)* if