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Lower central series of Artin–Tits and surface braid groups

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Abstract

We consider algebraic and topological generalisations of braid groups and pure braid groups, namely Artin–Tits groups (of spherical type) and surface (pure) braid groups, and we determine their lower central series and related residual properties.

Keywords: Braid groups; Surface braid groups; Artin–Tits groups; Mapping class groups; Lower central series

1. Introduction

Braid groups are ubiquitous objects. They appear in many different settings and may be defined in several equivalent ways, each allowing possible generalisations. In other words, braid groups are at the intersection of several families of groups. This paper deals with combinatorial properties of algebraic and topological generalisations of braid groups, called Artin–Tits groups (of spherical type) and surface braid groups respectively. In particular, we focus on their lower

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central series and related residual properties. Before stating the main results of the paper, we recall the definitions of such groups and of some classical notions in combinatorial group theory.

Artin–Tits groups

Let (W, S) be a Coxeter system, and let $m_{s,t}$ denote the order of the element st in W (for $s, t \in S$). Let B_W be the group defined by the following group presentation:

$$B_W = \langle S \mid \underbrace{st\cdots}_{m_{s,t}} = \underbrace{ts\cdots}_{m_{s,t}} \text{ for any } s \neq t \in S \text{ with } m_{s,t} < +\infty \rangle.$$

The group B_W is the *Artin–Tits group* associated to W. The group B_W is said to be of *spherical type* if W is finite. The kernel of the canonical projection of B_W onto W is called the *pure Artin–Tits group* associated to W.

Classical braid groups correspond to Artin–Tits groups of type $\mathcal{A}$. Artin–Tits groups (of spherical type) have been widely studied during the last few years, and several results on braid groups have been generalised to these groups, in particular their linearity.

Surface braid groups

Surface braid groups are a natural topological generalisation of braid groups and of fundamental groups of surfaces. They were first defined by Zariski during the 1930s (braid groups on the sphere had been considered earlier by Hurwitz), were re-discovered by Fox during the 1960s, and were used subsequently in the study of mapping class groups.

We recall the definition of surface braid groups as equivalence classes of geometric braids. In Section 5.2, we shall give a second equivalent definition using mapping class groups (braid groups were also defined by Fox in terms of fundamental groups of configuration spaces, see for instance [B,GG1,GG2]).

Let Σ be a connected, oriented surface. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ be a set of n distinct points (*punctures*) in the interior of Σ. A *geometric braid* on Σ based at $\mathcal{P}$ is a collection $(\psi_1, \ldots, \psi_n)$ of n disjoint paths (called *strands*) in $\Sigma \times [0, 1]$ which are monotone with respect to $t \in [0, 1]$, and satisfy $\psi_i(0) = (p_i, 0)$ and $\psi_i(1) \in \mathcal{P} \times \{1\}$. Two braids are considered to be equivalent if they are isotopic relative to the base point $\mathcal{P}$. The usual product of paths induces a group structure on the set of equivalence classes of braids. This group, which does not depend on the choice of $\mathcal{P}$, is called the *braid group* on n strands of the surface Σ, and shall be denoted by $B_n(\Sigma)$. A braid is said to be *pure* if $\psi_i(1) = (p_i, 1)$ for all $i = 1, \ldots, n$. The set of pure braids form a group called the *pure braid group* on n strands of the surface Σ, and shall be denoted by $P_n(\Sigma)$.

In the case of the disc $\mathbb{D}^2$, the group $B_n(\mathbb{D}^2)$ is isomorphic to the classical braid group B_n.

Lower central series and residual properties

Given a group G, we recall that the *lower central series* of G is the filtration $G := \Gamma_1(G) \supseteq \Gamma_2(G) \supseteq \cdots$, where $\Gamma_i(G) = [G, \Gamma_{i-1}(G)]$ for $i \geqslant 2$. The group G is said to be *perfect* if $G = \Gamma_2(G)$. From the lower central series of G, one defines another filtration $D_1(G) \supseteq D_2(G) \supseteq \cdots$, by setting $D_1(G) = G$, and for $i \geqslant 2$, defining $D_i(G) = \{x \in G \mid x^n \in \Gamma_i(G) \text{ for some } n \in \mathbb{N}^*\}$. This filtration was first considered by Stallings [S], and was later termed the *rational lower central series* of G [GLe].