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Symmetric subalgebras noncohomologous to zero in a complex semisimple Lie algebra and restrictions of symmetric invariants

Gang Han

Department of Mathematics, Zhejiang University, Hang Zhou 310027, China

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Abstract

Let $(\mathfrak{g},\theta)$ be a semisimple involutory Lie algebra and $(\mathfrak{g},\mathfrak{k})$ the corresponding symmetric pair. Let $\mathfrak{h}$ be a fundamental Cartan subalgebra of $(\mathfrak{g},\theta)$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. The semisimple involutory Lie algebras $(\mathfrak{g},\theta)$ with the symmetric subalgebra $\mathfrak{k}$ noncohomologous to zero in $\mathfrak{g}$ are completely classified by showing that $\mathfrak{k}$ is noncohomologous to zero in $\mathfrak{g}$ if and only if the *Spin* ν representation of $(\mathfrak{g},\theta)$ is primary. Based on this result we then determine the image of the restriction map $S(\mathfrak{h}^*)^{W(\mathfrak{g},\mathfrak{h})}\to S(\mathfrak{t}^*)^{W(\mathfrak{k},\mathfrak{t})}$, where $W(\mathfrak{g},\mathfrak{h})$ and $W(\mathfrak{k},\mathfrak{t})$ are the respective Weyl groups.

Keywords: Involutory semisimple Lie algebra; Symmetric subalgebra; Noncohomologous to zero; *Spin* ν representation; Symmetric invariants

1. Introduction

1.1. Let $\mathfrak{g}$ be a finite-dimensional reductive complex Lie algebra and $\mathfrak{k}$ be a reductive Lie subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g},\mathfrak{k})$ is called a reductive Lie subalgebra pair. Let $H(\mathfrak{g})$ and $H(\mathfrak{k})$ be the respective cohomology algebras of $\mathfrak{g}$ and $\mathfrak{k}$. The reductive subalgebra $\mathfrak{k}$ is called noncohomologous to zero in $\mathfrak{g}$ (or simply $\mathfrak{k}$ is n.c.z. in $\mathfrak{g}$) if the canonical map $H(\mathfrak{g})\to H(\mathfrak{k})$ is surjective. Let $H(\mathfrak{g}/\mathfrak{k})$ be the cohomology algebra of the pair $(\mathfrak{g},\mathfrak{k})$ defined as usual. For example see Chapter 10 of [2] or Section 2.2 of this paper for the definition of $H(\mathfrak{g}/\mathfrak{k})$. Let $H_0(\mathfrak{g}/\mathfrak{k})\subseteq H(\mathfrak{g}/\mathfrak{k})$ be a char-

E-mail address: mathhgg@zju.edu.cn.

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acteristic factor and P be the Samelson subspace of $(\mathfrak{g},\mathfrak{k})$. Then $H(\mathfrak{g}/\mathfrak{k}) \cong H_0(\mathfrak{g}/\mathfrak{k}) \otimes \wedge P$. If $\mathfrak{k}$ is noncohomologous to zero in $\mathfrak{g}$ then $H_0(\mathfrak{g}/\mathfrak{k}) \cong \mathbb{C}$, $H(\mathfrak{g}/\mathfrak{k}) \cong \wedge P$ and $H(\mathfrak{g}) \cong H(\mathfrak{g}/\mathfrak{k}) \otimes H(\mathfrak{k})$. So it will be nice if one can classify reductive pairs $(\mathfrak{g},\mathfrak{k})$ with $\mathfrak{k}$ noncohomologous to zero in $\mathfrak{g}$. We will solve this problem in the case that $\mathfrak{k}$ is a symmetric subalgebra in $\mathfrak{g}$, that is, $\mathfrak{k}$ is the fixed point of some involutory automorphism of $\mathfrak{g}$. If θ is an involutory automorphism of $\mathfrak{g}$, then the pair $(\mathfrak{g},\theta)$ is called an involutory Lie algebra.

Assume that $(\mathfrak{g},\theta)$ is a semisimple involutory Lie algebra (i.e., $\mathfrak{g}$ is semisimple). Let $\mathfrak{k}$ and $\mathfrak{p}$ be respectively the 1 and -1 eigenspace of θ. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called the Cartan decomposition of $(\mathfrak{g},\theta)$ and $(\mathfrak{g},\mathfrak{k})$ is called the symmetric pair of $(\mathfrak{g},\theta)$. Let

$$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a} \tag{1.1}$$

be a fundamental Cartan subalgebra of $(\mathfrak{g},\theta)$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{a}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{p}$. The projection $\mathfrak{h} \to \mathfrak{t}$ defined by (1.1) induces an inclusion $\mathfrak{t}^* \to \mathfrak{h}^*$, so we will always regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$. Let $\Delta(\mathfrak{g},\mathfrak{h})$ (respectively $\Delta(\mathfrak{g},\mathfrak{t})$, $\Delta(\mathfrak{k},\mathfrak{t})$) be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (respectively of $\mathfrak{g}$ with respect to $\mathfrak{t}$, of $\mathfrak{k}$ with respect to $\mathfrak{t}$). Let W be the Weyl group of $\Delta(\mathfrak{g},\mathfrak{h})$ acting on $\mathfrak{h}^*$. Let $W(\mathfrak{g},\mathfrak{t})$ and $W(\mathfrak{k},\mathfrak{t})$ be respectively the Weyl groups of $\Delta(\mathfrak{g},\mathfrak{t})$ and $\Delta(\mathfrak{k},\mathfrak{t})$ acting on $\mathfrak{t}^*$. Then $W(\mathfrak{k},\mathfrak{t})$ is a subgroup of $W(\mathfrak{g},\mathfrak{t})$. Let W_θ be the subgroup of W consisting of elements of W commuting with θ. One knows in [4] that $W(\mathfrak{g},\mathfrak{t})$ and W_θ are isomorphic.

Let $Spin\,\nu : \mathfrak{k} \to End\,S$ be the composition of the isotropy representation $\nu : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ with the spin representation $Spin : \mathfrak{so}(\mathfrak{p}) \to End\,S$. The map $Spin\,\nu$ is a representation of $\mathfrak{k}$ on S. The following theorem is the first main result in this paper and is proved in Theorem 3.1 and Proposition 3.3.

Theorem 1.1. *Let $(\mathfrak{g},\theta)$ be a semisimple involutory Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{a}$ is the centralizer of $\mathfrak{t}$ in $\mathfrak{p}$. Let $l = \dim \mathfrak{a}$. Then $\dim H(\mathfrak{g}/\mathfrak{k}) = 2^l |W(\mathfrak{g},\mathfrak{t})/W(\mathfrak{k},\mathfrak{t})|$ and $\dim H_0(\mathfrak{g}/\mathfrak{k}) = |W(\mathfrak{g},\mathfrak{t})/W(\mathfrak{k},\mathfrak{t})|$. In particular, the symmetric subalgebra $\mathfrak{k}$ is noncohomologous to zero in $\mathfrak{g}$ if and only if $W(\mathfrak{g},\mathfrak{t}) = W(\mathfrak{k},\mathfrak{t})$, if and only if the $Spin\,\nu$ representation of $(\mathfrak{g},\theta)$ is primary.*

Let $(\mathfrak{g},\theta) = (\mathfrak{b}, I) \oplus (\mathfrak{g}',\theta')$ be the decomposition of $(\mathfrak{g},\theta)$ into a trivial semisimple involutory Lie algebra $(\mathfrak{b}, I)$ and a reduced semisimple involutory Lie algebra $(\mathfrak{g}',\theta')$. Let $(\mathfrak{g}',\theta') = \bigoplus_{i=1}^n (\mathfrak{g}_i,\theta_i)$ be the decomposition of $(\mathfrak{g}',\theta')$ into reduced and irreducible semisimple involutory Lie algebras. Then the symmetric subalgebra $\mathfrak{k}$ is noncohomologous to zero in $\mathfrak{g}$ if and only if the symmetric subalgebra $\mathfrak{k}_i$ is noncohomologous to zero in $\mathfrak{g}_i$ for all i.

Note that the symmetric pairs of reduced and irreducible semisimple involutory Lie algebras with primary $Spin\,\nu$ representations have been classified in Theorem 4.13 of [3]. So the theorem means that semisimple involutory Lie algebras $(\mathfrak{g},\theta)$ with the symmetric subalgebra $\mathfrak{k}$ n.c.z. in $\mathfrak{g}$ are completely classified.

If $(\mathfrak{g},\theta)$ is a reductive involutory Lie algebra, then $(\mathfrak{g},\theta) = (\mathfrak{g}_0,\theta_0) \oplus (\tilde{\mathfrak{g}},\tilde{\theta})$ with $\mathfrak{g}_0$ the center of $\mathfrak{g}$ and $\tilde{\mathfrak{g}} = [\mathfrak{g},\mathfrak{g}]$. Let $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$ be respectively the θ-invariants of $\mathfrak{g}$ and $\tilde{\theta}$-invariants of $\tilde{\mathfrak{g}}$. By Lemma 3.2 one knows that $\mathfrak{k}$ is n.c.z. in $\mathfrak{g}$ if and only if $\tilde{\mathfrak{k}}$ is n.c.z. in $\tilde{\mathfrak{g}}$. Thus, given a reductive involutory Lie algebra $(\mathfrak{g},\theta)$, the symmetric subalgebra $\mathfrak{k}$ is n.c.z. in $\mathfrak{g}$ if and only if $(\mathfrak{g},\theta)$ can be written as the direct sum of an abelian involutory Lie algebra and a semisimple involutory Lie algebra $(\tilde{\mathfrak{g}},\tilde{\theta})$ with the symmetric subalgebra $\tilde{\mathfrak{k}}$ n.c.z. in $\tilde{\mathfrak{g}}$.