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On the blocks of semisimple algebraic groups and associated generalized Schur algebras

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Abstract

In this paper we give a new proof for the description of the blocks of any semisimple simply connected algebraic group when the characteristic of the field is greater than 5. The first proof was given by Donkin and works in arbitrary characteristic. Our new proof has two advantages. First we obtain a bound on the length of a minimum chain linking two weights in the same block. Second we obtain a sufficient condition on saturated subsets π of the set of dominant weights which ensures that the blocks of the associated generalized Schur algebra are simply the intersection of the blocks of the algebraic group with the set π. However, we show that this is not the case in general for the symplectic Schur algebras, disproving a conjecture of Renner.

Keywords: Blocks; Semisimple algebraic groups; Schur algebras

1. Introduction and notation

Let G be a semisimple simply connected algebraic group over an algebraically closed field k. We are interested in the category of all G-modules. When the field k has characteristic zero, this category is semisimple, in other words every G-module splits as a direct sum of simple modules. Over a field of characteristic $p > 0$, the category of G-modules is no longer semisimple. Nevertheless, it can be split into 'blocks' such that every G-module can be written as a direct

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sum of indecomposable (but not necessarily simple) modules belonging to these blocks. Thus in order to study the category of G-modules it is enough to study each block separately.

We recall the basic definitions and notation needed here. More details can be found in [10, Part II].

Let T be a maximal torus of G and let $W = N_G(T)/T$ be the Weyl group. Let B be a Borel subgroup containing T. Let $X = X(T)$ be the weight lattice and fix a non-singular, symmetric positive definite W-invariant form on $X \otimes_{\mathbb{Z}} \mathbb{R}$, denoted by $\langle .\,,.\rangle$. Let R be the root system, R^+ the set of positive roots which makes B the negative Borel and let S be the set of simple roots. Define the set of dominant weights by

$$X^+ = \{\lambda \in X \mid \langle \lambda, \check{\alpha} \rangle \geqslant 0 \ \forall \alpha \in S\}$$

where $\check{\alpha} = 2\alpha/\langle \alpha, \alpha \rangle$ for $\alpha \in R$. For $r \geqslant 1$ define also the set of p^r-restricted weights X_r by

$$X_r = \{\lambda \in X^+ \mid \langle \lambda, \check{\alpha} \rangle < p^r \ \forall \alpha \in S\}.$$

The weight lattice has a natural partial ordering: for $\lambda, \mu \in X$ we write $\lambda \geqslant \mu$ if and only if $\lambda - \mu$ is a sum of simple roots. Let w_0 be the longest element in the Weyl group W. We denote by β_0 the highest short root of R and by ρ half the sum of the positive roots. For each root $\beta \in R^+$ and each integer m, define the (affine) reflection $s_{\beta,m}$ on $X \otimes_{\mathbb{Z}} \mathbb{R}$ by

$$s_{\beta,m}(\lambda) = \lambda - \big(\langle \lambda, \check{\beta} \rangle - m\big)\beta.$$

Define the affine Weyl group W_p to be the group generated by all reflections $s_{\beta,mp}$ for $\beta \in R^+$, $m \in \mathbb{Z}$. Similarly, for any positive integer r we define W_{p^r} to be the group generated by all s_{β,mp^r} with $\beta \in R^+$ and $m \in \mathbb{Z}$. In this paper we always consider the dot action $w \cdot \lambda = w(\lambda + \rho) - \rho$ of W_p on X or $X \otimes_{\mathbb{Z}} \mathbb{R}$. So we view $s_{\beta,mp}$ as a reflection through the hyperplane

$$\{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\beta} \rangle = mp\}.$$

This action of the affine Weyl group W_p on $X \otimes_{\mathbb{Z}} \mathbb{R}$ defines a system of facets. A facet is a non-empty subset of the form

$$F = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \ \forall \alpha \in R_0^+(F),\ (n_\alpha - 1)p < \langle \lambda + \rho, \check{\alpha} \rangle < n_\alpha p \ \forall \alpha \in R_1^+(F)\}$$

for some $n_\alpha \in \mathbb{Z}$ and some disjoint decomposition $R^+ = R_0^+(F) \cup R_1^+(F)$. The closure $\bar{F}$ of F is equal to

$$\bar{F} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \ \forall \alpha \in R_0^+(F),\ (n_\alpha - 1)p \leqslant \langle \lambda + \rho, \check{\alpha} \rangle \leqslant n_\alpha p \ \forall \alpha \in R_1^+(F)\}$$

and the upper closure $\hat{F}$ of F is equal to

$$\hat{F} = \{\lambda \in X \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \lambda + \rho, \check{\alpha} \rangle = n_\alpha p \ \forall \alpha \in R_0^+(F),\ (n_\alpha - 1)p < \langle \lambda + \rho, \check{\alpha} \rangle \leqslant n_\alpha p \ \forall \alpha \in R_1^+(F)\}.$$