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Quasi-Lie structure of σ-derivations of $\mathbb{C}[t^{\pm 1}]$

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Abstract

Hartwig, Larsson and Silvestrov in [J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra 295 (2) (2006) 314–361] defined a bracket on σ-derivations of a commutative algebra. We show that this bracket preserves inner derivations, and based on this obtain structural results providing new insights into σ-derivations on Laurent polynomials in one variable.

Keywords: Quasi-Lie algebras; σ-derivations; Twisted bracket; q-deformed Witt algebras

1. Introduction

In [16,23–25] a new class of algebras called quasi-Lie algebras and its subclasses, quasi-hom-Lie algebras and hom-Lie algebras, have been introduced. An important characteristic feature of those algebras is that they obey some deformed or twisted versions of skew-symmetry and Jacobi identity with respect to some possibly deformed or twisted bilinear bracket multiplication. Quasi-Lie algebras include color Lie algebras, and in particular Lie algebras and Lie superalgebras, as well as various interesting quantum deformations of Lie algebras. Let us mention here as significant examples deformations of the Heisenberg Lie algebra, oscillator algebras, sl_2 and of other finite-dimensional Lie algebras, of infinite-dimensional Lie algebras of Witt and Virasoro type

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applied in physics within the string theory, vertex operator models, quantum scattering, lattice models and other contexts, as well as various algebras arising in connection to non-commutative geometry (see [3–14,16–21,26–29,34] and references therein). Many of these quantum deformations of Lie algebras can be shown to play the role of underlying algebraic objects for calculi of twisted, discretized or deformed derivations and difference type operators and thus in corresponding general non-commutative differential calculi.

Considering these deformed differential calculi, vector fields are replaced by twisted vector fields when derivations are replaced by twisted derivations. In [16, Theorem 5], it was proved that under some general assumptions these twisted vector fields are closed under a natural twisted non-associative skew-symmetric multiplication satisfying a twisted 6-term Jacobi identity. This identity generalizes the usual Lie algebras 3-term Jacobi identity that is recovered when no twisting is present (see Theorem 2.2.3). This result is shown to be instrumental to the construction of various examples and classes of quasi-Lie algebras. Both known and new one-parameter and multi-parameter deformations of Witt and Virasoro algebras and other Lie and color Lie algebras have been constructed within this framework in [16,23–25].

In this article, we gain further insight in the particular class of quasi-Lie algebra deformations of the Witt algebra. These were introduced in [16] via the general twisted bracket construction, and associated with twisted discretization of derivations generalizing the Jackson q-derivatives to the case of twistings by general endomorphisms of Laurent polynomials. In Section 2 we present necessary definitions, facts and constructions on σ-derivations that are central for this article. In Proposition 2.3.1 we observe that inner derivations are stable under the bracket defined in [16]. In the last part of this section, we present a characterization of the set of inner derivations for UFD (Proposition 2.4.1), and also general inclusions concerning sets of inner derivations and image and pre-image subsets with respect to the twisted bracket (Proposition 2.4.6). In Section 3 we develop the preceding framework for a particular important UFD, the algebra $\mathcal{A} = \mathbb{C}[t^{\pm 1}]$ of Laurent polynomials in one variable. For this specialization more deep and precise results can be obtained. We shall focus here on the fact that in the present paper we deal with an endomorphism σ of $\mathcal{A}$ which is NOT assumed to be an automorphism. The space of σ-derivations $\mathcal{D}_\sigma(\mathcal{A})$ endowed with the twisted bracket mentioned above is the deformation of the Witt algebra within the class of quasi-hom-Lie algebras in the sense of [16]. In Theorem 3.2.1 we show that the space of σ-derivations can be decomposed into a direct sum of the space of inner σ-derivations and a finite number of one-dimensional subspaces. In Theorem 3.3.4, we show that for arbitrary σ the $\mathbb{Z}$-gradation of this non-linearly q-deformed Witt algebra with coefficients in $\mathbb{C}$ becomes a $\mathbb{Z}/d\mathbb{Z}$-gradation with coefficients in $\mathbb{C}[T^{\pm 1}]$ for some element T of $\mathcal{A}$. The usual q-deformed Witt algebra associated to ordinary Jackson q-derivative corresponds to the automorphism $\sigma : t \mapsto qt$, and appears as a "limit case" where $d = 0$ and all σ-derivations are inner. In Subsection 3.4, we provide a more detailed description of what relations for the bracket in the non-linearly q-deformed Witt algebra become modulo inner σ-derivations. Finally, in Subsection 3.5, we describe normalizer-like subsets in detail for the non-linearly q-deformed Witt algebra.

Throughout this article, $\mathcal{A}$ will denote an associative, commutative and unital algebra over the field $\mathbb{C}$ of complex numbers. We will sometimes mention more general results concerning non-commutative algebras, and we will precise our assumptions on $\mathcal{A}$ in these cases. In the last section the algebra $\mathcal{A}$ will be the algebra of Laurent polynomials $\mathbb{C}[t^{\pm 1}]$.