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Automorphisms of toroidal Lie superalgebras

Dimitar Grantcharov [a,*], Arturo Pianzola [b,c,1]

[a] Department of Mathematics, San Jose State University, San Jose, CA 95192-0103, USA
[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
[c] Instituto Argentino de Matemática, Saavedra 15, (1083) Buenos Aires, Argentina

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Abstract

We give a description of the algebraic group $\mathbf{Aut}(\mathfrak{g})$ of automorphisms of a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ over an algebraically closed field k of characteristic 0, which is obtained by viewing $\mathfrak{g}$ as a module over a Levi subalgebra of its even part. As an application, we give a detailed description of the group of automorphism of the k-Lie superalgebra $\mathfrak{g} \otimes_k R$ for a large class of commutative rings R.

Keywords: Automorphisms; Toroidal superalgebras

1. Introduction

The group $\mathbf{Aut}(\mathfrak{g})$ of automorphisms of a finite-dimensional simple Lie superalgebra over an algebraically closed field k of characteristic 0 has been described in [S] and [GP]. The automorphisms of the finite-dimensional contragredient Lie superalgebras have been classified in [FSS1] and [vdL]. The results herein somehow refine and complement those of [S] and [GP], and provide a framework whereby the explicit nature of the abstract group of R-points of $\mathbf{Aut}(\mathfrak{g})$ can be determined for a large class of interesting rings R. In the Lie algebra case, the group $\mathbf{Aut}(\mathfrak{g})$ is a split extension of a finite constant group (the symmetries of the Dynkin diagram) by a simple group (the adjoint group, which is also the connected component of the identity of $\mathbf{Aut}(\mathfrak{g})$). By

* Corresponding author.
E-mail addresses: grantcharov@math.sjsu.edu (D. Grantcharov), a.pianzola@math.ualberta.ca (A. Pianzola).
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contrast, in the super case the analogous extension is not split, and the connected component of the identity of $\mathbf{Aut}(\mathfrak{g})$ need not even be reductive (let alone simple).

Our approach is to view $\mathfrak{g}$ as a module over a Levi subalgebra $\mathfrak{g}_0^{ss}$ of the even part of $\mathfrak{g}$. We will introduce three subgroups of $\mathbf{Aut}(\mathfrak{g})$; denoted by $\mathbf{Aut}(\mathfrak{g};\mathfrak{g}_0^{ss})$, $\mathbf{Aut}(\mathfrak{g},\Pi_0)$ and $\mathbf{H}$, which help clarify the nature of $\mathbf{Aut}(\mathfrak{g})$ and its outer part. These subgroups are interesting on their own right, and should prove useful in any future classification of multiloop algebras of $\mathfrak{g}$ via Galois cohomology (see [P4] and [P5] for the case of affine Lie algebras, and [GP] for the case of affine Lie superalgebras. See also [GiPi1] and [GiPi2] for toroidal Lie algebras).

As another application, we give an explicit description of the group of automorphisms of the (in general infinite-dimensional) k-Lie superalgebra $\mathfrak{g}\otimes_k R$ for a large class of commutative ring extensions R/k. This result generalizes the simple Lie algebra situation studied in [P1] (which is considerably easier by comparison). Of particular interest is the "toroidal case," namely when $R=k[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$.

2. Notation and conventions

Throughout k will be an algebraically closed field of characteristic zero. The category of associative commutative unital k-algebras will be denoted by *k-alg*. If V is a vector space over k, and R an object of *k-alg*, we set $V_R=V(R):=V\otimes_k R$. For a nilpotent Lie algebra $\mathfrak{a}$, a finite-dimensional $\mathfrak{a}$-module M, and $\lambda\in\mathfrak{a}^*$, we denote by M^λ the subspace of M on which $a-\lambda(a)$ acts nilpotently for every $a\in\mathfrak{a}$. We have then $M=\bigoplus_{\lambda\in\mathfrak{a}^*}M^\lambda$.

In what follows, $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ will denote a simple finite-dimensional Lie superalgebra over k (see [K] and [Sch] for details). A *Cartan subsuperalgebra* $\mathfrak{h}=\mathfrak{h}_{\bar 0}\oplus\mathfrak{h}_{\bar 1}$ of $\mathfrak{g}$, is by definition a selfnormalizing nilpotent subsuperalgebra. Then $\mathfrak{h}_{\bar 0}$ is a Cartan (in particular nilpotent) subalgebra of $\mathfrak{g}_{\bar 0}$, and $\mathfrak{h}_{\bar 1}$ is the maximal subspace of $\mathfrak{g}_{\bar 1}$ on which $\mathfrak{h}_{\bar 0}$ acts nilpotently (see Proposition 1 in [PS] for the proof). We denote by $\Delta=\Delta_{(\mathfrak{g},\mathfrak{h})}$ the *roots of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$. Thus $\Delta=\{\alpha\in\mathfrak{h}_{\bar 0}^*,\ \alpha\neq 0\mid\mathfrak{g}^\alpha\neq 0\}$. For $\bar\imath\in\mathbb{Z}/2\mathbb{Z}$ we set $\Delta_{\bar\imath}=\{\alpha\in\mathfrak{h}_{\bar 0}\mid\mathfrak{g}_{\bar\imath}^\alpha\neq 0\}$. Then $\Delta=\Delta_{\bar 0}\cup\Delta_{\bar 1}$. The *root lattice* $\mathbb{Z}\Delta$ of $(\mathfrak{g},\mathfrak{h})$ will be denoted by $Q_{(\mathfrak{g},\mathfrak{h})}$.

A linear algebraic group $\mathbf{G}$ over k (in the sense of [B]) can be thought as a smooth affine algebraic group (in the sense of [DG]) via its functor of points $\mathrm{Hom}_k(k[\mathbf{G}],-)$. We will find both of these points of view useful, and will henceforth refer to them simply as "Algebraic Groups" (and trust that the reader will be able at all times to understand which of these two viewpoints is being taken).

Let $\mathrm{Aut}_k(\mathfrak{g})$ be the (abstract) group of automorphisms of $\mathfrak{g}$. We point out that by definition, all automorphisms of a Lie superalgebra preserve the given $\mathbb{Z}/2\mathbb{Z}$-grading. It is clear that $\mathrm{Aut}_k(\mathfrak{g})$ gives rise to a linear algebraic group over k, which we denote by $\mathbf{Aut}(\mathfrak{g})$, whose functor of points is given by $\mathbf{Aut}(\mathfrak{g})(R)=\mathrm{Aut}_R(\mathfrak{g}_R)$; the automorphisms of the R-Lie superalgebra $\mathfrak{g}_R=\mathfrak{g}\otimes_k R$.

We will make repeated use of the following affine k-groups: $\mathbf{Hom}(U,V):R\to\mathrm{Hom}_{R\text{-mod}}(U\otimes R,V\otimes R)$ where U and V are finite-dimensional k-spaces, $\mathbf{G}_a:R\to(R,+)$, and $\mathbf{G}_m:R\to R^\times$ (the units of R). In addition we will also use many of the classical groups $\mathbf{GL}$, $\mathbf{SL}$, etc.; as well as the groups $\boldsymbol{\mu}_n$ defined by $\boldsymbol{\mu}_n(R):=\{r\in R\mid r^n=1\}$.

Recall that there are three types of simple finite-dimensional Lie superalgebras.[2] We use the notation of [Pen].

[2] These types are not mutually exclusive, and some overlap is indeed present in small rank.