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Mackey functors, induction from restriction functors and coinduction from transfer functors

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Abstract

Boltje's plus constructions extend two well-known constructions on Mackey functors, the fixed-point functor and the fixed-quotient functor. In this paper, we show that the plus constructions are induction and coinduction functors of general module theory. As an application, we construct simple Mackey functors from simple restriction functors and simple transfer functors. We also give new proofs for the classification theorem for simple Mackey functors and semisimplicity theorem of Mackey functors.

Keywords: Mackey functors; Restriction functors; Transfer functors; Plus constructions; Fixed-point functor; Induction; Coinduction; Simple Mackey functors

1. Introduction

The theory of Mackey functors was introduced by Green to provide a unified treatment of group representation theoretic constructions involving restriction, conjugation and transfer. Thévenaz and Webb improved Green's definition of a Mackey functor, and they realized Mackey functors as representations of the *Mackey algebra* $\mu_R(G)$. Using this identification, Thévenaz and Webb applied methods of module theory to classify the simple Mackey functors [11] and to describe the structure of Mackey functors [12]. Their description of simple Mackey functors

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used induction and inflation from subgroups and two dual constructions, known as the fixed-point functor and the fixed-quotient functor.

Applying the notion of Mackey functors to the problem of finding an explicit version of Brauer's induction theorem, Boltje introduced the theory of canonical induction [6,7]. In order to solve the problem in this general context, Boltje considered not only the category $\mathrm{Mack}_R(G)$ of Mackey functors, but also two more categories, namely the category $\mathrm{Con}_R(G)$ of conjugation functors and the category $\mathrm{Res}_R(G)$ of restriction functors. His main tools were the lower-plus and the upper-plus constructions, which extend the fixed-quotient and the fixed-point functors, respectively.

The lower-plus construction, denoted by $-_+$, is defined as a functor $\mathrm{Res}_R(G) \to \mathrm{Mack}_R(G)$. By introducing the *restriction algebra* $\rho_R(G)$, written ρ when R and G are understood, we realize the restriction functors as representations of the restriction algebra. This leads us to

Theorem 5.1. *The functors $-_+$ and ind_ρ^μ are naturally equivalent.*

On the other hand, the upper-plus construction, denoted by $-^+$, is defined as a functor $\mathrm{Con}_R(G) \to \mathrm{Mack}_R(G)$. By introducing the *transfer algebra* $\tau_R(G)$, written τ, and its representations, called *transfer functors* and realizing conjugation functors as representations of the *conjugation algebra* $\gamma_R(G)$, written γ, we prove

Theorem 5.2. *The functors $-^+$ and $\mathrm{coind}_\tau^\mu \mathrm{inf}_\gamma^\tau$ are naturally equivalent.*

As a consequence of these identifications, we realize the fixed-point and fixed-quotient functors as coinduced and induced modules, respectively. Given an RG-module V, we denote by FQ_V the fixed-quotient functor and by FP_V the fixed-point functor.

Proposition 5.4. *Let V be an RG-module. Then, the following isomorphisms hold.*

$$\text{(i)}\quad \mathrm{FQ}_V \simeq \mathrm{ind}_\rho^\mu \mathrm{inf}_\gamma^\rho D_V \quad \textit{and} \quad \text{(ii)}\quad \mathrm{FP}_V \simeq \mathrm{coind}_\tau^\mu \mathrm{inf}_\gamma^\tau D_V$$

where D_V denotes the γ-module which is non-zero only at the trivial group and $D_V(1) = V$.

We also prove that the Brauer quotient (also known as the bar construction) is the composition of certain restriction and deflation functors (see Corollary 5.7). Via this identification, we see that Thévenaz' twin functor is the composition of coinduction, inflation, deflation and restriction functors.

The plus constructions are also used by Bouc [4] and Symonds [9]. To obtain information about projective Mackey functors, Bouc considered restriction functors defined only on p-subgroups and also the functor $-_+$ (which is denoted by $\mathcal{I}$ in [4]). In [9], Symonds constructed induction formulae using the plus constructions described in terms of the zero degree group homology and group cohomology functors.

The subalgebra structure of the Mackey algebra, we describe above, leads us to

Theorem 3.2 *(Mackey structure theorem). The τ–ρ-bimodule ${}_\tau\mu_\rho$ is isomorphic to $\tau \otimes_\gamma \rho$.*

As a consequence of this theorem, we obtain several equivalences relating the functors between the algebras μ, τ, ρ and γ. Using some of these equivalences, we show that the well-