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Irredundant intersections of valuation overrings of two-dimensional Noetherian domains ☆

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Abstract

Let D be a two-dimensional Noetherian domain, let R be an overring of D, and let Σ and Γ be collections of valuation overrings of D. We consider circumstances under which $(\bigcap_{V\in\Sigma} V)\cap R = (\bigcap_{W\in\Gamma} W)\cap R$ implies that $\Sigma = \Gamma$. We show that if R is integrally closed, these representations are "strongly" irredundant, and every member of $\Sigma\cup\Gamma$ has Krull dimension 2, then $\Sigma = \Gamma$. If in addition Σ and Γ are Noetherian subspaces of the Zariski–Riemann space of the quotient field of D (e.g. if Σ and Γ have finite character), then the restriction that the members of $\Sigma\cup\Gamma$ have Krull dimension 2 can be omitted. An example shows that these results do not extend to overrings of three-dimensional Noetherian domains.

Keywords: Valuation theory; Noetherian domains

1. Introduction

If H is an integral domain with quotient field F, then an *overring* of H is a ring R such that $H \subseteq R \subseteq F$. If also R is a valuation ring (that is, the ideals of R are linearly ordered with respect to $\subseteq$), then R is a *valuation overring* of H. This article is motivated by the problem of describing integrally closed overrings of a two-dimensional Noetherian domain D. Non-Noetherian integrally closed overrings of D arise for example when considering rings of invariants, affineness of open sets of projective schemes, holomorphy rings and direct limits of blowup algebras. Since a domain is integrally closed if and only if it is the intersection of its valuation overrings, a funda-

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mental question arising in this context is whether for valuation overrings $V_1, \dots, V_n, W_1, \dots, W_m$ of D and an integrally closed overring R of D, the finite cancellation property holds:

$$V_1 \cap \dots \cap V_n \cap R = W_1 \cap \dots \cap W_m \cap R \quad \Rightarrow \quad \{V_1, \dots, V_n\} = \{W_1, \dots, W_m\}.$$

To make this question meaningful, we must assume that none of the V_i's or W_j's can be omitted from the intersection; that is, they are *irredundant* in their respective intersections. W. Heinzer and J. Ohm proved in Corollary 6.4 of [11] that if all the V_i's and W_j's have Krull dimension 1, then cancellation does indeed hold. In fact, this result, which we discuss in more detail in Section 5, follows from general principles, and it holds for any underlying domain D, not just the Noetherian case.

However, returning to our setting of overrings of the two-dimensional Noetherian domain D, it is easy to see cancellation can fail if one of the V_i's has Krull dimension > 1:

Example 1.1. Let H be an integrally closed local Noetherian domain of Krull dimension 2. Let P be a fixed height 1 prime ideal of H, and let X be the collection of height 1 prime ideals of H distinct from P. Let $R = \bigcap_{Q \in X} H_Q$, and observe that $H \neq R$ since H is a Krull domain. Let W be a valuation overring of H of Krull dimension 2 such that $H \subseteq W \subseteq H_P$. Then $H = W \cap R = H_P \cap R$ and W and H_P are irredundant in these representations of H, yet $W \neq H_P$.

Inspection of this example shows that cancellation fails for a rather trivial reason; namely, the valuation ring W can be replaced in the intersection by the larger valuation ring H_P. Thus we rephrase the cancellation property to require that none of the V_i's or W_j's can be replaced in the intersection with one of its proper overrings. We say in this case that the V_i's and W_j's are *strongly irredundant* in the intersection. In Theorem 5.4 we prove that cancellation holds for any (possibly infinite) collections of valuation overrings of D, when these valuation rings all have Krull dimension 2 and the intersections are strongly irredundant. (What we prove is actually stronger than this.) If in addition we assume that these collections of valuation rings are Noetherian subspaces of the Zariski–Riemann space of the quotient field of D, then we need not restrict to valuation overrings of Krull dimension 2 (Corollary 5.6). Thus, as a consequence, the finite cancellation property holds for strongly irredundant intersections over D. Example 6.2 shows that this theorem is tight, in the sense that the cancellation property can fail over a Noetherian domain of Krull dimension 3. We in fact use heavily throughout this paper that the base domain D is a Noetherian ring of Krull dimension 2.

The cancellation problem can be rephrased in terms of the uniqueness of representations of an integrally closed overring of D, a point of view we elaborate on now by introducing some terminology. The *Zariski–Riemann space* of the domain H is the set $\mathrm{Zar}(H)$ of all valuation overrings of H endowed with the topology whose basic open sets are of the form

$$U(x_1, \dots, x_n) := \{V \in \mathrm{Zar}(H)\colon\ x_1, \dots, x_n \in V\},$$

where $x_1, \dots, x_n$ are in the quotient field of H; cf. [20, Chapter VI, Section 17]. Let R be an overring of H. We say that a collection Σ of valuation overrings of H is an *R-representation of H* if $H = (\bigcap_{V \in \Sigma} V) \cap R$.

An R-representation Σ of H is *irredundant* if no proper subset of Σ is an R-representation of H. An R-representation Σ of H is *strongly irredundant* if no member V of Σ can be replaced with a proper overring V_1 of V. More precisely, Σ is a strongly irredundant R-representation of