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Journal of Algebra 300 (2006) 316–338
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Towards a finite presentation for the Nottingham group

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Received 17 November 2005
Available online 27 March 2006
Communicated by Eamonn O'Brien

Dedicated to Charles Leedham-Green on the occasion of his 65th birthday

Abstract

This paper exhibits 2 generators for the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ for $p > 3$ and $\frac{p+5}{2}$ relations that they satisfy modulo terms of the lower central series, and begins an investigation of the properties of an infinite pro-p group $\mathcal{G}$ with a pair of generators satisfying these relations. The ultimate aim is to show that the graded Lie algebras of $\mathcal{G}$ and $\mathcal{N}(\mathbb{F}_p)$ with respect to their lower central series are isomorphic.

Keywords: Group theory; Pro-p groups

Introduction

The Nottingham group $\mathcal{N} = \mathcal{N}(\mathbb{F}_p)$, where p is prime, is the group of $\mathbb{F}_p$-algebra automorphisms of $\mathbb{F}_p[\![t]\!]$ that centralise $(t)/(t^2)$. These groups have been studied in some detail since 1990, and it is well known that $\mathcal{N}$ is an hereditarily just infinite pro-p group, and that it is universal, in the sense that it contains a copy of every countably based pro-p group, see [1]. It has more recently been proved by Ershov to be finitely presented (as pro-

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.02.028

p group), for odd p, see [3], where it is shown that if p is odd then, $\mathcal{N}$ has a 2-generator n-relator presentation with $n \leqslant 12p + 32$.

The impetus for seeking a finite presentation came originally from the ideas of Caranti in [2]. He exhibited a 2-generator presentation for the graded Lie algebra $L(\mathcal{N})$ of $\mathcal{N}$ with respect to the lower central series, and used this to show that $L(\mathcal{N})$ does not have a finite presentation, but has a central extension that does. This was used by Ershov in his proof that $\mathcal{N}$ is finitely presented.

Taking the presentation in [2] as evidence in this direction, Leedham-Green and O'Brien conjectured that $\mathcal{N}$ has a 2-generator 5-relator presentation. They were able to produce strong computational evidence suggesting that 5 relations is sufficient by looking at quotients by terms of the lower central series in a group with such a presentation.

It is well known that the lower central factors $\gamma_i(\mathcal{N})/\gamma_{i+1}(\mathcal{N})$ have order p except when $i \equiv 1 \bmod (p-1)$, in which case they have order p^2. The Lie algebra M considered by Caranti has a natural grading in which M_i has order p except when $i \equiv 1 \bmod (p-1)$ or $i \equiv -1 \bmod (p-1)$ in which case it has order p^2.

In this paper we take $\mathcal{G}$ to be a an infinite pro-p group with a 2-generator $\frac{p+5}{2}$-relator presentation satisfying certain conditions (see Section 1), based on Caranti's presentation for M and such that $\mathcal{N}$ has a pair of generators satisfying these conditions. Caranti's proof can then be used to show that the lower central factors of $\mathcal{G}$ have order p except when $i \equiv 1 \bmod (p-1)$, in which case they have order p^2, and possibly when $i \equiv -1 \bmod (p-1)$ in which case they have order dividing p^2. The question to be answered is how to prove that the factors $\gamma_i(G)/\gamma_{i+1}(G)$ for $i \equiv -1 \bmod (p-1)$ in fact have order p. If this can be proved, then it will imply that $L(\mathcal{G})$ is isomorphic to $L(\mathcal{N})$ for all such $\mathcal{G}$, and that $\mathcal{N}$ has a 2-generator $\frac{p+5}{2}$-relator presentation.

The fact that the graded Lie algebra $L(\mathcal{N})$ is not finitely presented whereas the pro-p group $\mathcal{N}$ is finitely generated stems from the fact that in $L(\mathcal{N})$ only information from the quotients $\gamma_i(\mathcal{N})/\gamma_{i+1}(\mathcal{N})$ and $[\gamma_i(\mathcal{N}), \gamma_j(\mathcal{N})]/\gamma_{i+j+1}(\mathcal{N})$ is available. Thus, much of information from the group theory is lost. The basic idea of this project is to use Caranti's proof as a basis and extend his ideas to use deeper information about commutators and pth powers (see Section 1). This paper explores how this might be done.

A strong motivation for seeking to prove this result is that it would be a step towards a 'characterisation' of $\mathcal{N}$, see [4, pp. 314–315]. Such a characterisation should specify a small value of i with the property that if G is a pro-p group with $G/\gamma_i(G) \cong \mathcal{N}/\gamma_i(\mathcal{N})$ then $G \cong \mathcal{N}$.

If this projected finite presentation is found to lead to a group with $L(\mathcal{G})$ isomorphic to $L(\mathcal{N})$, then it will follow that, $G/\gamma_{2p+1}(G) \cong \mathcal{N}/\gamma_{2p+1}(\mathcal{N})$ implies that $L(G) \cong L(\mathcal{N})$. Proving that G is, in fact, isomorphic to $\mathcal{N}$ would, of course be a much harder problem.

At the present time it seems that a proof for the case $p = 5$ is almost complete. This starts from the material in this paper, taking the relations to greater depth, and strengthens the results here. Some details of this work are given at the end of Section 5.

It has been essential to be able to carry out very involved calculations in $\mathcal{N}$ in order to make any progress on this project, and we have used a 'Nottingham group calculator' provided by Leedham-Green extensively, both to make conjectures and to check commutator calculations for small values of p.