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Quantization of branching coefficients for classical Lie groups

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Abstract

We study natural quantizations K of branching coefficients corresponding to the restrictions of the classical Lie groups to the Levi subgroups of their standard parabolic subgroups. The polynomials obtained can be regarded as generalizations of Lusztig q-analogues of weight multiplicities. For GL_n they coincide with Poincaré polynomials previously studied by Shimozono and Weyman. They also appear in the Hilbert series of the Euler characteristic of certain graded virtual G-modules and, by a result of Broer, admit nonnegative coefficients providing that restrictive conditions are verified. When the Levi subgroup considered is isomorphic to a direct product of linear groups, we prove that these polynomials admit a stable limit $\tilde{K}$ which decomposes as nonnegative integer combination of Poincaré polynomials. For a general Levi subgroup, it is conjectured that the polynomials K have nonnegative coefficients when they are parametrized by two partitions.

When $G = GL_n$, the polynomials K can be interpreted as quantizations of the Littlewood–Richardson coefficients. We show that there also exists a duality between tensor product coefficients for types B, C, D (defined as the analogues of the Littlewood–Richardson coefficients) and branching coefficients corresponding to the restriction of SO_{2n} to subgroups defined from orthogonal decompositions of the root system D_n (which are not Levi subgroups). These coefficients can also be quantified but the q-analogues obtained may have negative coefficients. Given a tensor product Π of irreducible GL_N-modules, we then introduce for each classical group $G = SO_N$ or Sp_N some q-analogues $\mathfrak{D}$ of the multiplicities obtained by decomposing Π into its G-irreducible components. We establish a duality between the polynomials $\mathfrak{D}$ and $\tilde{K}$. According to a conjecture by Shimozono, the stable one-dimensional sums for nonexceptional affine crystals are expected to coincide with the polynomials $\mathfrak{D}$ associated to a sequence of rectangular partitions of decreasing heights.

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1. Introduction

The Kostka coefficients and the Littlewood–Richardson coefficients which have many occurrences in the representation theory of GL_n admit interesting q-analogues. Given λ a partition of length at most n and $\mu \in \mathbb{Z}^n$, the q-analogue of the Kostka coefficient $K_{\lambda,\mu}$ giving the dimension of the weight space μ in the irreducible finite dimensional GL_n-module $V^{GL_n}(\lambda)$ of highest weight λ is the Kostka–Foulkes polynomial $K_{\lambda,\mu}(q)$ (also called Lusztig q-analogue of weight multiplicity). Consider $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ an r-tuple of partitions of lengths summing to n and denote by μ the n-tuple obtained by reading the parts of the $\mu^{(p)}$'s from left to right. There exist in the literature different quantizations of the Littlewood–Richardson coefficient $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ giving the multiplicity of $V^{GL_n}(\lambda)$ in the tensor product $V^{GL_n}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(r)})$.

In [9] Lascoux, Leclerc and Thibon have introduced such a q-analogue by mean of certain generalizations of semi-standard Young tableaux called ribbon tableaux. They have proved in [10] that the polynomials obtained belong to a family of parabolic Kazhdan–Lusztig polynomials introduced by Deodhar which have nonnegative integer coefficients [6].

When $\mu^{(1)}, \ldots, \mu^{(r)}$ are rectangular partitions, it is also possible to define q-analogues of the coefficients $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ by considering the one-dimensional sums $X^{\emptyset}_{\lambda,\boldsymbol{\mu}}(q)$ obtained from affine $A^{(1)}_{n-1}$-crystals associated to Kirillov–Reshetikhin $U'_q(\widehat{sl_n})$-modules [5].

Consider $\eta = (\eta_1, \ldots, \eta_r)$ a sequence of positive integers summing to n and suppose that $\mu^{(p)}$ has length η_p for any $p = 1, \ldots, r$. The Littlewood–Richardson coefficient $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ also coincides with the multiplicity of the tensor product $V^{GL_{\eta_1}}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_{\eta_r}}(\mu^{(r)})$ in the restriction of $V^{GL_n}(\lambda)$ to its Levi subgroup $GL_\eta = GL_{\eta_1} \times \cdots \times GL_{\eta_r}$. This duality permits to express $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ in terms of a Kostant-type partition function. By quantifying this partition function, Shimozono and Weyman [19] have introduced another natural q-analogue of $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$ that we will denote by $K^{GL_n,I}_{\lambda,\mu}(q)$ (I being the set of the simple roots of GL_η). The polynomials $K^{GL_n,I}_{\lambda,\mu}(q)$ are Poincaré polynomials and appear in the Hilbert series of the Euler characteristic of certain graded virtual G-modules. By a result of Broer [1], they admit nonnegative coefficients providing that the $\mu^{(p)}$'s are rectangular partitions of decreasing heights. In this case Shimozono has proved in [16] that $K^{GL_n,I}_{\lambda,\mu}(q)$ coincide with $X^{\emptyset}_{\lambda,\boldsymbol{\mu}}(q)$. This result which is based on a combinatorial description of the polynomials $K^{GL_n,I}_{\lambda,\mu}(q)$, permits in particular to recover that they have nonnegative coefficients independently of the results of Broer. Under the same hypothesis, it is conjectured that $K^{GL_n,I}_{\lambda,\mu}(q)$ also coincides with the LLT quantization of $c^{\lambda}_{\mu^{(1)},\ldots,\mu^{(r)}}$. When the $\mu^{(p)}$'s are simply row partitions, we have $\mu = (\mu^{(1)}, \ldots, \mu^{(r)}) \in \mathbb{Z}^n$ and $K^{GL_n,I}_{\lambda,\mu}(q)$ is the Kostka–Foulkes polynomial associated to the weights λ and μ.

Let G be one of the classical groups GL_n, SO_{2n+1}, Sp_{2n} or SO_{2n} and R^+_G its set of positive roots. Kostka and Littlewood–Richardson coefficients can be regarded as branching coefficients corresponding to the restriction of GL_n to its principal Levi subgroups. This naturally yields to study the branching coefficients corresponding to the restriction to a subgroup G_0 (not necessarily of Levi type) of G and their corresponding q-analogues. The branching coefficients which