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Baer sums in homological categories

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Abstract

We give a unified treatment of the Baer sums in the context of efficiently homological categories which, on the one hand, contains any category of groups with multiple operators and more generally any semi-abelian variety and, on the other hand, the category of Hausdorff groups and more generally any category of semi-abelian Hausdorff algebras. This gives rise to a generalized "Euclide's Postulate" and a five terms exact sequence.

Keywords: Baer sum; Protomodular, additive and homological categories; Topological group; Topological semi-abelian algebra

Introduction

On the one hand, the notion of homological (i.e. pointed protomodular and regular) category is a context in which it is possible to deal, in full generality, with the notion of exact sequence, and it was shown in [2,9] that any homological category $\mathbb{C}$ satisfies the "passive" homological machinery: namely when a diagram satisfies some homological hypothesis, it satisfies also the homological conclusions; see, for instance, the Noether isomorphisms, the short five, 3×3 and "snake" lemmas. But this notion is unable in general to produce the "active" part of homology, namely to produce new exact sequences from given ones, as it is the case, for instance, in the category *Gp* of groups with the calculation of the Baer sum of two exact sequences having same abelian kernel and producing the same action on this kernel.

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On the other hand, there was described in [7], for any finitely complete Barr exact category $\mathbb{C}$, a very general description of the Baer sum process concerning the objects X having a global support in $\mathbb{C}$ and endowed with an autonomous Mal'cev operation $p : X \times X \times X \to X$: actually, with any object of this kind, there is associated an abelian group object $d(X)$ in $\mathbb{C}$ which defines a "direction" functor $d : Mal\,\mathbb{C}_g \to Ab\,\mathbb{C}$, in the same way as with any nonempty K-affine space is associated a K-vector space. This direction functor d is a cofibration whose any fibre is a groupoid canonically endowed with a closed symmetric tensor product which "is" the Baer sum. Indeed, if you consider a group C and the slice category $\mathbb{C} = Gp/C$ of groups above C (which is a Barr exact category), an object $g : G \to C$ is abelian in $\mathbb{C}$ if and only if it is a group homomorphism with abelian kernel [2]. It has a global support if and only if it is surjective. The "direction" of such an extension g with abelian kernel A is then nothing but the projection $C \ltimes_\phi A \to C$ whose domain is the semidirect product given by the classical group action $\phi : C \to Aut\,A$ associated with this extension (in other words, its direction is its associated C-module). In this context, the symmetric tensor product on such extensions is nothing but the classical Baer sum, and the abelian group structure of the isomorphism classes of such extensions is nothing but $Opext(C, A, \phi)$ as in [26] for instance.

More generally, when a category $\mathbb{C}$ is protomodular [6] (as this is still the case for $\mathbb{C} = Gp/C$), any object X has at most one Mal'cev operation which is necessarily autonomous; so that the existence of such an operation becomes a property and is no longer a further structure. In those circumstances, the object X in question is said to be abelian in $\mathbb{C}$. Accordingly, if the category $\mathbb{C}$ is both protomodular and Barr exact, the subcategory (actually subgroupoid) of abelian objects with global support and direction A inherits certainly a closed symmetric monoidal structure (= Baer sum), and the set of isomorphism classes of such objects inherits an abelian group structure, which (via Corollary 4 in [7]) is nothing but the first cohomology group $H^1_{\mathbb{C}}(A)$ of $\mathbb{C}$ with coefficients in A (in the sense of [1] for instance).

The first aim of this work, as the previous remark about the category $\mathbb{C} = Gp/C$ makes us hope, is to show that the description, recalled in the second paragraph above, of the abstract Baer sum defined in any Barr exact context can be greatly simplified when the ground category $\mathbb{C}$ in question is moreover protomodular, and can be reduced to a classical scheme concerning the exact sequences. This produces a unified treatment of the Baer theory in the context of any semi-abelian variety [13] and in particular of any category of groups with multiple operators [21]. This is the root of the classically known (but unexplained up to now) parallelism of treatment of homology theory for groups and Lie R-algebras (see, for instance, [17,22,25] for some classical illustrations of this parallelism, and [20] for a less classical one). More generally, these processes lead to a generalized "Euclide's Postulate" and, on the model of the homology of groups or Lie R-algebras, to a five terms exact sequence.

Actually this simplified description does hold, with no extra charge, in the wider context of *efficiently homological* categories (see Definition 1.1) which allows us to enlarge the class of examples dealing with Baer theory, including the categories *AbTop*, *AbHaus*, *GpTop* and *GpHaus* of topological and Hausdorff (abelian) groups and more generally any category of semi-abelian topological and Hausdorff algebras in the sense of [3].

Certainly, the pioneering work in aiming a unified treatment of the Baer sums goes back to Gerstenhaber [18]. But there were no detailed proofs (for instance, on p. 63, of the main fact that $i \vee -j$ is a kernel map), and, more importantly, the context was much more restricted, see [8,27] for a precise comparison of that context with the protomodular one. On the other hand, our present work completes the attempt of [14].