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Lattice-ordered algebras with a d-basis

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Abstract

A d-basis for a lattice-ordered algebra is a vector lattice basis in which each element is a d-element. In this paper we study the structures of lattice-ordered algebras with a d-basis.

Keywords: d-Basis; Lattice-ordered algebra

1. Introduction

M. Henriksen posed the following problem in [5, Problem 4].

"Develop a structure theory for a class of lattice-ordered rings that include semigroup algebras over $\mathbb{R}$ ordered as follows. If S is a multiplicative semigroup, $s_1, s_2, \ldots, s_n \in S$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$, let $\sum a_i s_i \geqslant 0$ if $a_i \geqslant 0$ for $1 \leqslant i \leqslant n$. Do this at least for a class of semiproups large enough to include $\{1, x, \ldots, x^n, \ldots\}$ and the semigroup of unit of matrices $\{E_{ij}\}$ (where E_{ij} has a 1 in row i and column j, and zeros elsewhere for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$)."

The purpose of the present paper is to introduce a class of lattice-ordered algebras (ℓ-algebras) that include matrix algebras and polynomial algebras, with the lattice order defined above, as well as other well-known examples in lattice-ordered rings (ℓ-rings). We consider the structure for this class of ℓ-algebras. The ℓ-algebras under study are those with a d-basis that is a basis for

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a vector lattice in which each element is a d-element (Definition 2.1). In Section 2 we provide some basic properties and examples of such ℓ-algebras. In Section 3 we consider the structure of such ℓ-algebras in which the identity element is basic. The Sections 4 and 5 are devoted to the commutative case and to the ℓ-reduced case, respectively.

We first review some definitions and results in ℓ-rings. The reader is referred to G. Birkhoff [2] for the general theory of lattices, and G. Birkhoff and R.S. Pierce [3] and A. Bigard, K. Keimel, and S. Wolfenstein [1] for the general theory of ℓ-rings. Throughout this paper F always denotes a totally ordered subfield of the totally ordered field $\mathbb{R}$ of real numbers.

Let R be an ℓ-ring. The *positive cone* of R is defined as $R^+ = \{r \in R \mid r \geqslant 0\}$. The elements in R^+ are called *positive*. A nonzero element a in R^+ is called *strictly positive*, denoted by $a > 0$. If $x \in R$, then the *positive part*, *negative part*, and *absolute value* of x is defined as $x^+ = x \vee 0$, $x^- = -x \vee 0$, and $|x| = x \vee -x$, respectively. An *ℓ-algebra* A over F is an ℓ-ring and an algebra over F such that $F^+A^+ \subseteq A^+$. An ℓ-algebra is called *unital* if it has an identity element, and called *ℓ-unital* if the identity element is positive. An ℓ-ring is called an *ℓ-domain* if $a > 0$ and $b > 0$ imply $ab > 0$, and an ℓ-ring is called *ℓ-reduced* if it does not contain nonzero positive nilpotent elements. An element a in an ℓ-ring is called *basic* if $a > 0$ and, for any $b, c \geqslant 0$, $a \geqslant b, c \geqslant 0$ implies that b and c are comparable, that is, $b \geqslant c$ or $c \geqslant b$. Two elements x, y in an ℓ-ring are called *disjoint* if $x, y \geqslant 0$ and $x \wedge y = 0$. A set $\{x_i \mid i \in I\}$ in an ℓ-ring is called *disjoint* if each $x_i > 0$ and any two different elements in the set are disjoint. It is well known that a disjoint set in an ℓ-algebra is a linearly independent set. An element e in an ℓ-ring R is called an *f-element* of R if $b \wedge c = 0$ implies $eb \wedge c = be \wedge c = 0$ and e is called a *d-element* of R if $b \wedge c = 0$ implies $eb \wedge ec = be \wedge ce = 0$ for all $b, c \in R$. A subset S of an ℓ-ring is called *convex* if $0 \leqslant r \leqslant s$ and $s \in S$ imply $r \in S$. An *ℓ-ideal* of an ℓ-algebra is a ring ideal and a linear subspace which is also a convex sublattice. An ℓ-algebra A is called *ℓ-simple* if 0 and A are the only ℓ-ideals of A, and A is called *ℓ-semisimple* if the intersection of all of its maximal ℓ-ideals is zero. Let A be an ℓ-algebra and $M_1, \ldots, M_k$ be convex vector sublattices of A. Then $M_1 \oplus \cdots \oplus M_k$ denotes the direct sum of $M_1, \ldots, M_k$ as vector lattices. Let $I_1, \ldots, I_k$ be ℓ-ideals of A. Then $I_1 \oplus \cdots \oplus I_k$ coincide with the direct sum of $I_1, \ldots, I_k$ as ℓ-ideals.

Let A be an ℓ-algebra over F and let $E_F(A)$ be the algebra over F of all the endomorphisms of the vector space A over F. Define the positive cone $E_F(A)^+ = \{\theta \in E_F(A) \mid A^+\theta \subseteq A^+\}$. Then $(E_F(A), E_F(A)^+)$ becomes a partially ordered algebra (po-algebra) over F. For each $a \in A$, define $\theta_a \in E_F(A)$ by $x\theta_a = xa$ for each $x \in A$. Then the correspondence $a \to \theta_a$ is called the *right regular representation* of A as an algebra. The right regular representation is called an *ℓ-representation* if for each $a, b \in A$,

$$\theta_{a\vee b} = \theta_a \vee \theta_b \quad \text{and} \quad \theta_{a\wedge b} = \theta_a \wedge \theta_b.$$

The left regular representation of A could be defined similarly. An ℓ-algebra is called *regular* if its left and right regular representations are both ℓ-representations. The reader is referred to [2,3] for more information on regular ℓ-rings.

2. Basic properties and examples

In this section we give some basic properties and examples of unital ℓ-algebras with a d-basis.

Definition 2.1. Let A be an ℓ-algebra over F. A nonempty subset S of A is called a *d-basis* if the following three conditions are satisfied: