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The Bender method in groups of finite Morley rank

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0. Introduction

The algebraicity conjecture for simple groups of finite Morley rank, also known as the Cherlin–Zilber conjecture, states that simple groups of finite Morley rank are simple algebraic groups over algebraically closed fields. In the last 15 years, the main line of attack on this problem has been Borovik's program of transferring methods from finite group theory. Borovik's program has led to considerable progress; however, the conjecture itself remains decidedly open. In Borovik's program, groups of finite Morley rank are divided into four types, odd, even, mixed, and degenerate, according to the structure of their Sylow 2-subgroup. For *even* and *mixed type* the algebraicity conjecture has been proven. The present paper is part of the program to bound the *Prüfer rank* of minimal simple groups of finite Morley rank and *odd type*.

In [1], Cherlin and Jaligot achieved a bound of Prüfer rank two for *tame* minimal simple groups. Here a group of finite Morley rank is said to be tame if it does not involve a field of finite Morley rank with a proper infinite definable subgroup of its multiplicative group. Cherlin, Jaligot, and the present author will bound the Prüfer rank at two in [2].

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Tameness is used in two important ways in [1]. The final number theoretic contradiction of [1] uses tameness in an essential way, and [2] will completely replace this argument. However, the very first use of tameness in [1] produces the following fact, which shows that intersections of Borel subgroups are Abelian.

Jaligot's lemma. *(See [1, Lemma 3.11].) Let G be a* tame *minimal connected simple group of finite Morley rank. Let B_1 and B_2 be two distinct Borel subgroups of G with $O(B_1) \neq 1$ and $O(B_2) \neq 1$. Then $F(B_1) \cap F(B_2) = 1$.*

The present paper examines the worst violations of Jaligot's lemma in the *nontame* context, i.e., those involving non-Abelian intersections of Borel subgroups. We fail to exclude such non-Abelian intersections outright, but we gain much information about the specific local configuration responsible for non-Abelian intersections.

In the context of minimal simple groups, the present paper provides a analog of Bender's uniqueness theorem [3, Theorem 28.2] (see also [4, Chapter 5] and [5, Section 9]), a result underlying the Bender method [3, Section 28] of analyzing the maximal subgroups containing the centralizer of an involution. Both the Bender uniqueness theorem and the present paper provide information about the normalizers of various subgroups of the intersection of two distinct maximal subgroups. However, our situation will be simplified by two facts: torsion behaves extremely well (see Section 2), and our "torsion-free primes," so-called reduced ranks, are naturally ordered by their degree of unipotence (see Fact 1.16).

In [2], much of the information about this non-Abelian configuration, plus analysis of the relevant Abelian intersections, is used to prove the following.

Theorem. *Let G be a minimal connected simple group of finite Morley rank and odd type with a strongly embedded subgroup. Then G has Prüfer rank one.*

One proves the bound on Prüfer rank by showing that simple groups of finite Morley rank and Prüfer rank at least three have strongly embedded subgroups [6].

The bulk of this paper consists of the analysis of non-Abelian maximal intersections of Borel subgroups in a minimal connected simple group of finite Morley rank (see Section 3). A priori, the analysis of these maximal intersections yields only a limited description of nonmaximal intersections.

Proposition 4.1. *Let G be a minimal connected simple group of finite Morley rank. Let B_1, B_2 be two distinct Borel subgroups of G, and H a connected subgroup of the intersection $B_1 \cap B_2$. Then the following hold.*

(1) *H' is rank homogeneous for $r' := \bar{r}_0(H')$.*
(2) *Every connected nilpotent subgroup of H is Abelian.*
(3) *$F_{r'}(H) = U_{0,r'}(H)$ is a Sylow $U_{0,r'}$-subgroup of H.*
(*etc.*)

In high Prüfer rank, the experience of [2] suggests that Proposition 4.1 itself is insufficient, but that the results of Section 3 which describe the configuration arising from